Series 4

Exercise 1. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\{X_n\}_{n \geq 1}$ a sequence of real valued random variables $X_n: \Omega \to \mathbb{R}$ and $X: \Omega \to \mathbb{R}$ another random variable. We say that the sequence $X_n$ converges to $X$ almost surely ($X_n \to X$ a.s.) if the set $A$ defined as

$$A = \{ \omega : \lim_{n \to \infty} X_n(\omega) \neq X(\omega) \},$$

satisfies $P(A) = 0$. Show that

i) $X_n \to X$ in probability if and only if

$$\lim_{n \to \infty} \mathbb{E} \left[ \frac{|X_n - X|}{1 + |X_n - X|} \right] = 0,$$

ii) if $X_n \to X$ in probability then there exists a subsequence $n_k$ such that $X_{n_k} \to X$ a.s. for $k \to \infty$.

Hints: for i), choose without loss of generality $X = 0$. For ii), exploit i) and the fact that if $\sum_{i=1}^{\infty} \mathbb{E}[|Y_n|] < \infty$ then $\sum_{i=1}^{\infty} Y_i < \infty$ a.s. for any sequence $Y_n$ of random variables.

Remark. This property of convergent sequences of the random variables, together with the ones proved in Series 3, is part of the following scheme

$$X_n \to X \text{ a.s.} \quad \Rightarrow \quad X_n \to X \text{ in probability} \quad \Rightarrow \quad X_n \to X \text{ in distribution} \quad \Rightarrow \quad X_n \to X \text{ in } L^p$$

In particular, the dashed arrows are not always true, but

- $X_n \to X$ in probability $\implies X_{n_k} \to X$ a.s. for a subsequence (Exercise 1),
- $X_n \to X$ in probability, $|X_n| \leq Y$ for all $n$ and $Y \in L^p \implies X_n \to X$ in $L^p$,
- $X_n \to c \in \mathbb{R}$ in distribution $\implies X_n \to c$ in probability.

Exercise 2. (Conditional Jensen inequality) Suppose that $\phi: \mathbb{R} \to \mathbb{R}$ is Borel measurable and convex with $\mathbb{E} |\phi(X)| < \infty$, where $X: \Omega \to \mathbb{R}$ is a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. Show that

$$\phi \left( \mathbb{E}(X|\mathcal{G}) \right) \leq \mathbb{E}(\phi(X)|\mathcal{G}),$$

where $\mathcal{G} \subseteq \mathcal{F}$ is a $\sigma$-algebra.
**Exercise 3.** Let \((W(t), t \geq 0)\) be a one-dimensional Brownian motion. Show that

\[
P(W(t) \in B | W(s)) = \frac{1}{\sqrt{2\pi(t-s)}} \int_B e^{-\frac{|x-W(s)|^2}{2(t-s)}} \, dx \quad \text{a.s.}
\]

for all \(0 \leq s < t\) and all Borel set \(B \in \mathcal{B}\).

**Exercise 4.** (Brownian bridge) Consider the interval \([0, 1]\). A Brownian bridge is a standard Gaussian process \((Z(t), 0 \leq t \leq 1)\) such that \(\text{Cov}(Z(t), Z(s)) = \min\{s, t\} - st\).

Let \((W(t), 0 \leq t \leq 1)\) be a standard Brownian motion.

i) Show that \(Z(t) = W(t) - tW(1)\) is a Brownian bridge.

ii) Using the Brownian bridge, construct such a process that is Gaussian with \(E(Z(t)) = x - \frac{t}{1}(x - y)\) and \(\text{Cov}(Z(t), Z(s)) = \min\{s, t\} - \frac{st}{1}\).

iii) (Implementation) Write a Matlab code to simulate a Brownian bridge \((Z(t), 0 \leq t \leq 2)\) with \(Z(0) = 1, Z(2) = 2\) on a partition with \(\Delta t = 2^{-4}, 2^{-6}, 2^{-8}\). Approximate \(E(Z(t))\) over \(M = 20, 200, 2000\) trajectories.

**Exercise 5.** (Implementation exercise) For a given \(n \geq 1\), we approximate a Brownian motion as

\[
W_n(t) = \frac{2^{n+1}-1}{\sum_{k=0}^{2^{n+1}-1} s_k(t)\xi_k},
\]

where \(\{s_k\}_{k=0}^{2^{n+1}-1}\) are the Schauder functions defined in Series 3 and \(\{\xi_k\}_{k=0}^{2^{n+1}-1}\) is a sequence of independent \(N(0, 1)\) random variables. Let \(P = \{0 = t_0 < t_1 < \ldots < t_N = 1\}\) be the uniform partition of \([0, 1]\) with \(\Delta t = 2^{-12}\).

i) For \(n = 3, \ldots, 10\), compute \(W_n\) on the partition \(P\).

ii) Verify numerically that \(E(W_n(t)) = 0\).

iii) Observe numerically that the sequence \(V_n = \sum_{i=1}^{N} |W_n(t_i) - W_n(t_{i-1})|\) diverges.

iv) Consider the series of the time derivative of \(W\)

\[
D_n(t) = \frac{d}{dt}W_n(t) = \sum_{k=0}^{2^{n+1}-1} h_k(t)\xi_k.
\]

For \(n = 3, \ldots, 10\) compute \(D_n\) on \(P\) and observe that the series diverges.

---

General information and series on [https://anmc.epfl.ch](https://anmc.epfl.ch)