Series 1

Exercise 1. Consider a scalar standard Brownian motion (Wiener process) on [0, 1].

i) Write a Matlab code to simulate a discretized Brownian motion on $t_j = j\Delta t$ with

\[ \Delta t = 2^{-4}, 2^{-6}, 2^{-8}, \]

ii) Compute the mean over 20, 200, 2000 trajectories.

iii) Compute the discretized stochastic process

\[ X(t_j) = X_0 e^{(\lambda - \frac{1}{2} \mu^2) t_j + \mu W(t_j)} \]

for $\lambda = 2$, $\mu = 1$, $X_0 = 1$ with $\Delta t = 2^{-4}, 2^{-6}, 2^{-8}$.

iv) Compute the mean of $X(t)$ over 20, 200, 2000 trajectories. Can you guess what $\mathbb{E}(X(t))$ is?

Exercise 2. Let $\lambda = 2$, $\mu = 1$ and consider the stochastic differential equation

\[ dX(t) = \lambda X(t)dt + \mu X(t)dW(t) \quad 0 \leq t \leq T, \]

\[ X(0) = X_0. \] (1)

and the Euler–Maruyama (EM) method for (1)

\[ X_n = X_{n-1} + \lambda X_{n-1} \Delta t + \mu X_{n-1} (W(t_n) - W(t_{n-1})). \]

The exact solution of (1) is given by (see Exercise 1)

\[ X(t) = X_0 e^{(\lambda - \frac{1}{2} \mu^2) t + \mu W(t)} . \]

Compute a discretized Brownian path over [0, 1] with $\delta t = 2^{-8}$ and compare the exact solution (on the discretized path) with the EM method (using the same Brownian path) with $\Delta t = 2^p \delta t$, $2^q \delta t$.

Exercise 3. (Strong convergence)

A numerical method for an SDE

\[ dX(t) = f(X(t))dt + g(X(t))dW(t) \quad 0 \leq t \leq T, \]

\[ X(0) = X_0. \] (2)

is said to have a strong order of convergence equals to $r$ if there exists a constant $C$ such that

\[ \mathbb{E}|X_n - X(t_n)| \leq C(\Delta t)^r, \]

for any fixed $t_n = n\Delta t \in [0, T]$. For the SDE of Exercise 2 set $t_n = T = 1$, apply the EM method and set $e_n^\Delta := \mathbb{E}|X_n - X(t_n)|$. Verify numerically that $e_n^\Delta \leq C(\Delta t)^{1/2}$. To evaluate $\mathbb{E}|X_n - X(t_n)|$ you need to compute $\frac{1}{M} \sum_{i=1}^{M} |X_i^n - X^n(t_n)|$, i.e. the average over $M$ realizations of the random variables at time $t_n = 1$. For that:

i) take $M = 10000$ independent discretized Brownian path over [0, 1] with $\delta t = 2^{-9}$ ;

ii) for each path apply EM with $\Delta t = 2^p \delta t$, $0 \leq p \leq 4$ and store the endpoint error (at $t = T$) ;

iii) take the mean over the error and repeat the report the result ($\Delta t$ versus strong error) in a loglog plot.
Exercise 4. (Weak convergence)

A numerical method for an SDE

\[ dX(t) = f(X(t))dt + g(X(t))dW(t) \quad 0 \leq t \leq T \]
\[ X(0) = X_0. \] (3)

is said to have a weak order of convergence equals to \( r \) if there exists a constant \( C \) such that

\[ |E_p(X_n) - E_p(X(t_n))| \leq C(\Delta t)^r, \]

for any fixed \( t_n = n\Delta t \in [0, T] \) and all sufficiently smooth function \( p \). For the SDE of Exercise 2 with \( \lambda = 2, \mu = 0.1 \), set \( t_n = T = 1 \) and \( e_{\Delta t}^w := |E(X_n) - E(X(t_n))| \) and verify numerically that \( e_{\Delta t}^w \leq C\Delta t \).