Series 8

Exercise 1. Consider the system of differential equations
\[ \dot{y} = f(y), \quad y(0) = y_0, \]
and let \( \varphi_h(y_0) \) denote the associated flow at time \( t = h \). Let \( \Phi_h \) be a one step method of order \( p \) satisfying
\[ \Phi_h(y_0) = \varphi_h(y_0) + C(y_0)h^{p+1} + O(h^{p+2}). \]

i) Show that the adjoint method \( \Phi_h^* \) satisfies
\[ \Phi_h^*(y_0) = \varphi_h(y_0) + (-1)^p C(y_0)h^{p+1} + O(h^{p+2}). \]

ii) Show that the (maximal) order of a symmetric method is even.

Exercise 2.

i) Show that the adjoint of a collocation method based on the nodes \( c_1, c_2, \ldots, c_s \) is again a collocation method based on the nodes \( c_1^*, c_2^*, \ldots, c_s^* \), where \( c_i^* = 1 - c_{s+1-i} \), for \( i = 1, 2, \ldots, s \).

ii) Deduce from i) that the Gauss collocation methods are symmetric.

Exercise 3. Consider the Hamiltonian system
\[ \dot{y} = J^{-1}\nabla H(y) = f(y), \quad y(0) = y_0, \quad (1) \]
and its variational equation
\[ \dot{\Psi} = \frac{\partial f}{\partial y} \Psi, \quad \Psi(0) = I. \quad (2) \]

i) Show that \( \text{trace}\left(\frac{\partial f}{\partial y}\right) = 0 \) and hence that \( \det \Psi \) is a first integral of (2).

ii) Show that for every bounded open set \( \Omega \subset \mathbb{R}^d \) and for every \( t \in \mathbb{R} \), for which the flow \( \varphi_t \) of (1) exists, we have (Liouville’s theorem)
\[ \text{Vol}(\varphi_t(\Omega)) = \text{Vol}(\Omega), \]
where \( \text{Vol}(\Omega) = \int_{\Omega} dy \).

iii) More generally, show that the flow of a general system of differential equations \( \dot{y} = f(y) \) in \( \mathbb{R}^n \) is volume preserving if and only if \( \text{div} f(y) = 0 \) for all \( y \in \mathbb{R}^n \).

Exercise 4.

i) Prove that a smooth transformation \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) is symplectic if and only if it is volume and orientation preserving.

ii) Is this statement still true for a transformation \( g : \mathbb{R}^{2d} \to \mathbb{R}^{2d} \) with \( d > 1 \)? If not, find a counterexample.

General information and series on [http://anmc.epfl.ch/Numerical.html](http://anmc.epfl.ch/Numerical.html)