Series 2

Exercise 1.

i) By applying the mean value theorem to \( \int_{t_0}^{t_0 + h} f(s, y(s)) ds \), motivate the \( \theta \) method

\[
y_1 = y_0 + hf\left(t_0 + \theta h, y_0 + \theta (y_1 - y_0)\right),
\]

for the approximation of the initial value problem

\[
y'(t) = f(t, y(t)), \quad y(t_0) = y_0.
\]

Which methods do we obtain for \( \theta = 0 \), \( \theta = \frac{1}{2} \), and \( \theta = 1 \) ?

ii) Further, another variant of the \( \theta \) method is given by

\[
y_1 = y_0 + h(1 - \theta) f(t_0, y_0) + h\theta f(t_0 + h, y_1).
\]

Which methods do we obtain for \( \theta = 0 \), \( \theta = \frac{1}{2} \), and \( \theta = 1 \) ?

iii) Show that both versions (1) and (2) are Runge-Kutta methods (give the coefficients).

Exercise 2. Give the order of the methods defined by (1) and (2) in Exercise 1 for the values \( \theta = 0 \), \( \theta = \frac{1}{2} \), and \( \theta = 1 \). Further, show that the Runge method (see course) has order 2.

Exercise 3. Show that the order \( p \) of an \( s \)-stage explicit Runge-Kutta method satisfies \( p \leq s \).

Exercise 4. Derive the order conditions for the coefficients \( a_{ij}, b_i \), for \( i, j = 1, \ldots, s \), of a Runge-Kutta method such that the method has order \( p = 3 \).

Exercise 5. Show that, if the field \( f(t, y) \) satisfies a Lipschitz condition in \( y \), an implicit Runge-Kutta method applied to \( y'(t) = f(t, y(t)) \) is well defined (for a stepsize that is small enough), i.e., the nonlinear system defining the method has a unique solution.

Hint: use a fixed-point theorem.

Exercise 6. Consider a Runge-Kutta method with coefficients \( a_{ij}, b_i \), for \( i, j = 1, \ldots, s \). Suppose that the method has order \( p \) for all autonomous systems

\[
y'(t) = f(y(t)), \quad y(0) = y_0.
\]

Show that, if the coefficients \( c_i \) are defined by \( c_i = \sum_{j=1}^{s} a_{ij} \), for \( i = 1, \ldots, s \), this method has also order \( p \) when applied to a non autonomous system

\[
y'(t) = f(t, y(t)), \quad y(0) = y_0.
\]

Recall. For a consistent method, i.e., with order \( p \geq 1 \), we have \( \sum_{i=1}^{s} b_i = 1 \).

Hint: use the transformation \( Y(t) = \begin{pmatrix} t \\ y(t) \end{pmatrix} \)

General information and series on http://anmc.epfl.ch/Numerical.html