Series 12

Exercise 1. Consider a collocation method with \( s = 2 \) nodes. Discuss for which values of \( c_1, c_2 \) the method is A-stable.

Exercise 2. Consider the linear system \( y' = Ay \), where \( A \) is a complex \( n \times n \) matrix. We assume that

\[
\Re \langle y, Ay \rangle \leq 0 \quad \forall y \in \mathbb{C}^n, \tag{1}
\]

where \( \langle \cdot, \cdot \rangle \) denotes the canonical scalar product in \( \mathbb{C}^n \).

i) Show that any solution of \( y' = Ay \) has the property that \( \| y \| \) is a decreasing function in time.

ii) Let \( R(z) \) be a rational function that is bounded for \( \Re z \leq 0 \). Prove that in the norm

\[
\| C \| = \sup_{u, v \in \mathbb{C}^n, \| w \| \leq 1, \| v \| \leq 1} |\langle u, Cv \rangle| \quad \text{for } C \in \mathbb{C}^{n \times n},
\]

i.e., in the matrix norm associated to the scalar product, we have

\[
\| R(A) \| \leq \sup_{\Re z \leq 0} | R(z) |.
\]

iii) Let \( R(z) \) be the stability function of an A-stable Runge–Kutta method. Deduce that the numerical solution \( y_{n+1} = R(hA)y_n \) is contractive

\[
\| y_{n+1} \| \leq \| y_n \|.
\]

Hints for ii)

a) Consider first the case where \( A \) is a normal matrix, i.e., \( A \) satisfies \( AA^* = A^*A \). Recall, that a normal matrix can be diagonalized by a unitary matrix \( Q \), i.e., \( QQ^* = Q^*Q = I \).

b) For a general matrix \( A \), consider the matrix function

\[
A(w) = \frac{w}{2}(A + A^*) + \frac{1}{2}(A - A^*).
\]

Show that \( A(w) \) satisfies \( [1] \) for all \( \Re w \geq 0 \).

c) Show that for fixed vectors \( u, v \in \mathbb{C}^n \) the rational function \( \varphi(w) = \langle u, R(A(w))v \rangle \) has no poles in \( \Re w \geq 0 \).

d) Show that \( A(iy) \) is a normal matrix for any \( y \in \mathbb{R} \).

e) Prove that \( |\varphi(1)| \leq \sup_{y \in \mathbb{R}} |\varphi(iy)| \leq \sup_{\Re z \leq 0} | R(z) | \| u \| \| v \| .\)

Exercise 3. Let \( (k, j) \in \mathbb{N} \times \mathbb{N} \) be given.

i) Show that there exists a unique rational approximation \( \frac{P(z)}{Q(z)} \) to the exponential \( e^z \) of order \( k + j \) with \( \deg(P) = k, \deg(Q) = j \) and \( Q(0) = 1 \).
ii) Prove that the rational approximation $R(z)$ studied in [1] is explicitly given by $R_{kj}(z) = \frac{P_{kj}(z)}{Q_{kj}(z)}$ with

$$P_{kj}(z) = 1 + \frac{kz}{j + k} + \ldots + \frac{k(k - 1) \cdots 1}{(j + k) \cdots (j + 1) k!} z^k,$$

$$Q_{kj}(z) = 1 - \frac{jz}{k + j} + \ldots + (-1)^j \frac{j(j - 1) \cdots 1}{(k + j) \cdots (k + 1) j!} z^j.$$

**Hint:** Use Lemma 1 and Lemma 2 of the Chapter “Stability domains of collocation methods” of the course. Those provide results for rational functions of the type

$$R(z) = \frac{M^{(s)}(1) + zM^{(s-1)}(1) + \ldots + z^s M(1)}{M^{(s)}(0) + zM^{(s-1)}(0) + \ldots + z^s M(0)},$$

where $M(t)$ is a polynomial of degree $s$ and satisfies $M^{(s)}(t) = 1$. Try to find an appropriate polynomial $M(t)$ such that the numerator has degree $k$ and the denominator has degree $j$.

**Exercise 4.** Consider a numerical method $\Phi_h$ of order $p$ for systems of differential equations

$$\dot{y} = f(y), \quad y(t_0) = y_0.$$ Let $y_2 = \Phi_h \circ \Phi_h(y_0)$ and $\omega = \Phi_{2h}(y_0)$. Show that

$$y(t_0 + 2h) - y_2 = 2C(y_0) h^{p+1} + O(h^{p+2}), \quad \text{(2)}$$
$$y(t_0 + 2h) - \omega = C(y_0)(2h)^{p+1} + O(h^{p+2}), \quad \text{(3)}$$

where the same quantity $C(y_0)$ appears in both equations (2) and (3).