Chapter 3

Important classes of numerical methods

We present in this chapter some of the most important classes of numerical methods for systems of differential equations with initial condition of the form

\[ \begin{align*}
\dot{y} &= f(t, y), \quad t \in [0, T], \\
y(t_0) &= y_0,
\end{align*} \tag{3.1} \]

where \( f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) is a given smooth vector field, \( y(0) = y_0 \in \mathbb{R}^d \) is the given initial condition, \( T \) is the final time, and \( y : [0, T] \to \mathbb{R}^d \) is the solution of the system.

We shall focus on one step methods

\[ y_{n+1} = \Phi_h(y_n), \]

where \( h_n \) denotes the stepsize (which may be constant, \( h_n = h \), or may be variable at each step), to compute by induction a discrete approximation \( y_0, y_1, y_2, \ldots \) of the exact solution \( y(t_0), y(t_1), y(t_2), \ldots \) at times \( t_n \) defined as \( t_{n+1} = t_n + h_n \).

Given \( y_n \), the simplest method to compute \( y_{n+1} \approx y(t_{n+1}) \) where \( t_n = nh \) is obtained by substituting in (3.1) the derivative approximation \( \dot{y}(t_n) \approx \frac{y_{n+1} - y_n}{h} \), which yields the explicit Euler method

\[ y_{n+1} = y_n + hf(t_n, y_n). \]

3.1 Runge-Kutta methods

3.1.1 Definition

As presented before, the explicit Euler method

\[ y_{n+1} = y_n + hf(t_n, y_n). \]

Given \( t_0 \) and \( T \), we consider the Cauchy problem

Given a stepsize \( h \) such that \( T - t_0 = Nh \), we consider the time sequence \( t_0 < t_1 < \ldots < t_n \). The integration of (3.1) from \( t_n \) to \( t_n + h = t_{n+1} \) gives

\[ y(t_1) = y(t_0) + \int_{t_0}^{t_0+h} f(s, y(s)) \, ds. \]

The approximation of the integral in the second term of the right hand side leads to a numerical method.
Example 3.1.1. 1. (Euler, 1768) The approximation

\[ \int_{t_0}^{t_0+h} f(s, y(s)) \approx hf(t_0, y_0), \]

leads to the Euler explicit method.

2. (Cauchy, 1824) The approximation

\[ \int_{t_0}^{t_0+h} f(s, y(s)) \approx hf(t_0 + h, y(t_0 + h)), \]

leads to the Euler implicit method.

Definition 3.1.2. A one step method has order \( p \) if for all problems (3.1) with \( f(t, y) \) sufficiently differentiable we have, after one step,

\[ y_1 - y(t_0 + h) = O(h^{p+1}), \quad h \to 0, \]

where we start on the exact solution for the method, \( y_0 = y(t_0) \).

Example 3.1.3. For a smooth \( y \), we apply the Taylor expansion,

\[
y(t_0 + h) = y(t_0) + h\dot{y}(t_0) + \frac{h^2}{2} \ddot{y}(t_0) + ... \\
= y(t_0) + hf(t_0, y_0) + \frac{h^2}{2} \left( \frac{\partial f}{\partial t}(t_0, y_0) + \frac{\partial f}{\partial y}(t_0, y_0)f(t_0, y_0) \right) + ... \tag{3.3}
\]

Consider the Euler explicit method

\[ y_1 = y(t_0) + hf(t_0, y(t_0)). \]

Compared to (3.3) we see that \( y_1 - y(t_1) = O(h^2), \quad h \to 0 \), so that the method is of order 1.

Consider now the Euler implicit method

\[
y_1(h) = y(t_0) + hf(t_0 + h, y(t_0 + h)) = y(t_0) + h\phi(h), \quad y_1(0) = y(t_0).
\]

We expand \( \phi \) :

\[
\phi(h) = \phi(0) + \phi'(0)h + O(h^2), \\
\phi(h) = f(t_0, y(t_0)) + \left( \frac{\partial f}{\partial t}(t_0, y_0) + \frac{\partial f}{\partial y}(t_0, y_0)y'_1(h) \right)h + O(h^2) \\
= f(t_0, y(t_0)) + \left( \frac{\partial f}{\partial t}(t_0, y_0) + \frac{\partial f}{\partial y}(t_0, y_0)f(t_0, y_0) \right)h + O(h^2).
\]

Then we obtain that

\[ y_1 = y(t_0) + hf(t_0, y(t_0)) + h^2 \left( \frac{\partial f}{\partial t}(t_0, y_0) + \frac{\partial f}{\partial y}(t_0, y_0)f(t_0, y_0) \right), \]

which compared to (3.3) gives that \( y_1 - y(t_1) = O(h^2), \quad h \to 0 \), and the method is of order 1.
Example 3.1.4. Consider now the midpoint rule for quadrature formulas

\[
\int_{t_0}^{t_0+h} f(s, y(s)) \approx hf\left(t_0 + \frac{h}{2}, y\left(t_0 + \frac{h}{2}\right)\right).
\]

Here, we see that we have to approximate \(y(t_0 + \frac{h}{2})\). There are multiple ways to do it. First, we consider the method used by the mathematician Runge (1895):

\[
\begin{cases}
  y_{1/2} = y_0 + \frac{h}{2}f(t_0, y_0), \\
  y_1 = y_0 + hf\left(t_0 + \frac{h}{2}, y_{1/2}\right).
\end{cases}
\]

We observe that the first line is an Euler explicit step. We rewrite this method more elegantly

\[
\begin{cases}
  k_1 = f(t_0, y_0), \\
  k_2 = f\left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1\right), \\
  y_1 = y_0 + hk_1.
\end{cases}
\]

We can check that this method has order 2. A second way to approximate \(y(t_0 + \frac{h}{2})\), is to use an Euler implicit step in the first line,

\[
\begin{cases}
  y_{1/2} = y_0 + \frac{h}{2}f\left(t_0 + \frac{h}{2}, y_{1/2}\right), \\
  y_1 = y_0 + hf\left(t_0 + \frac{h}{2}, y_{1/2}\right),
\end{cases}
\]

which rewritten gives

\[
\begin{cases}
  k_1 = f\left(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1\right), \\
  y_1 = y_0 + hk_1.
\end{cases}
\]

This is the so called \textit{implicit midpoint rule}, which is of order 2 as well.

The generalization of the previous examples leads to a general family called Runge-Kutta methods.

Definition 3.1.5. (Runge, 1895; Kutta, 1901; Heun, 1900) Let \(s \in \mathbb{N}^*\) and let \(b_i, a_{ij}, c_i, 1 \leq i, j \leq s\) be real numbers. An \textit{s-stage Runge-Kutta method} for the integration of (3.1) is given by

\[
k_i = f\left(t_0 + c_i h, y_0 + h \sum_{j=1}^{s} a_{ij} k_j\right), \quad i = 1, ..., s, \tag{3.4}
\]

\[
y_1 = y_0 + h \sum_{i=1}^{s} b_i k_i. \tag{3.5}
\]

Remark:

1. We usually take \(c_i = \sum_{j=1}^{s} a_{ij}\). This ensures that all \(k_i\) are first order approximation, i.e. \(f(t_0 + c_i h, y(t_0 + c_i h)) = \mathcal{O}(h^2)\).

2. If \(a_{ij} = 0, \forall j \geq i\), the method is explicit. Otherwise it is implicit.
3. Usually, we display such a method with its characteristic Butcher tableau of coefficients,

\[
\begin{array}{c|cccc}
  c_1 & a_{11} & \ldots & a_{1s} \\
  \vdots & \vdots & & \vdots \\
  c_s & a_{s1} & \ldots & a_{ss} \\
\end{array}
\]

(3.6)

**Example 3.1.6.**

1. The Euler explicit method is given by

\[
k_1 = f(t_0, y_0),
\]

\[
y_1 = y(t_0) + hk_1.
\]

Its Butcher tableau is

\[
\begin{array}{cc}
  0 & 0 \\
  1 & 1 \\
\end{array}
\]

(3.7)

2. In the same way, the Butcher tableau of the Euler implicit method

\[
k_1 = f(t_0 + h, y_0 + hk_1),
\]

\[
y_1 = y(t_0) + hk_1,
\]

is given by

\[
\begin{array}{cc}
  1 & 1 \\
  1 & 1 \\
\end{array}
\]

(3.8)

3. Consider now the Runge method given in the Example 3.1.4. Its Butcher tableau is given by

\[
\begin{array}{ccc|ccc}
  0 & 0 & 0 \\
  1/2 & 1/2 & 0 \\
  & & 0 & 1 \\
\end{array}
\]

(3.9)

**Remark:**

1. Does an implicit method exist? If \( f \) is Lipschitz and \( h \) small enough, yes (see exercise serie 2).

2. A variable stepsize can be used:

\[
y_1 = y_0 + h_0 \sum_{i=1}^{s} b_i k_i(h_0)
\]

\[\vdots\]

\[
y_{k+1} = y_k + h_k \sum_{i=1}^{s} b_i k_i(h_k)
\]

3.1.2 Order theory

**Global convergence**

**Theorem 3.1.7.** Let \( y(t) \) be the solution of

\[
\begin{align*}
  y'(t) &= f(t, y(t)), & t & \in [t_0, T], \\
  y(t_0) &= y_0.
\end{align*}
\]

(3.10)

Assume that