Weak Cardinality Theorems for First-Order Logic

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Fundamentals of Computation Theory 2003

Outline

1. **History**
   - Enumerability in Recursion and Automata Theory
   - Known Weak Cardinality Theorem
   - Why Do Cardinality Theorems Hold Only for Certain Models?
2. **Unification by First-Order Logic**
   - Elementary Definitions
   - Enumerability for First-Order Logic
   - Weak Cardinality Theorems for First-Order Logic
3. **Applications**
   - A Separability Result for First-Order Logic

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Motivation of Enumerability

Problem

Many functions are not computable or not efficiently computable.

Example

- #SAT:
  How many satisfying assignments does a formula have?

Motivation of Enumerability

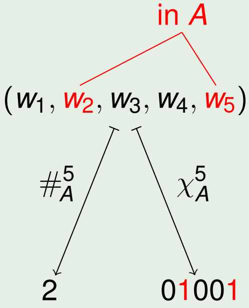
Problem

Many functions are not computable or not efficiently computable.

Example

For difficult languages A:

- Cardinality function $\#_A^n$:
  How many input words are in A?
- Characteristic function χ_A^n:
  Which input words are in A?



Motivation of Enumerability

Problem

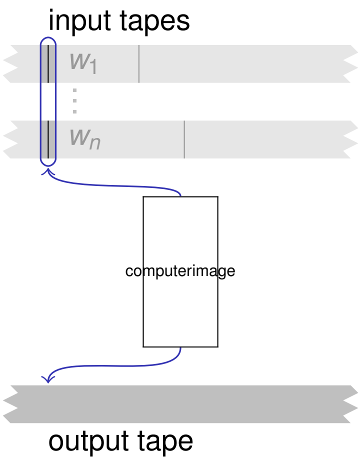
Many functions are not computable or not efficiently computable.

Solutions

Difficult functions can be

- computed using probabilistic algorithms,
- computed efficiently on average,
- approximated, or
- enumerated.

Enumerators Output Sets of Possible Function Values



Definition (1987, 1989, 1994, 2001)

An *m*-enumerator for a function f

1. reads n input words $w_1, \ldots, w_n$,
2. does a computation,
3. outputs at most m values,
4. one of which is $f(w_1, \ldots, w_n)$.

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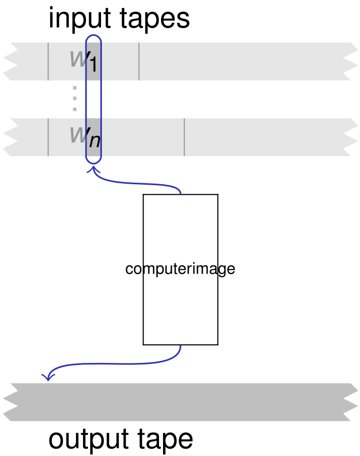

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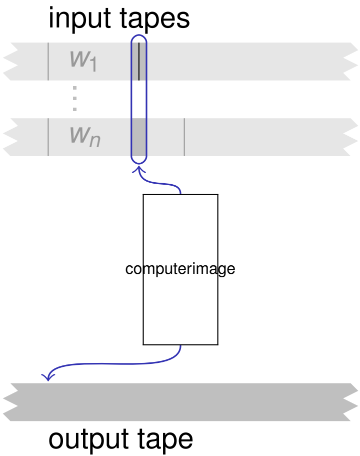

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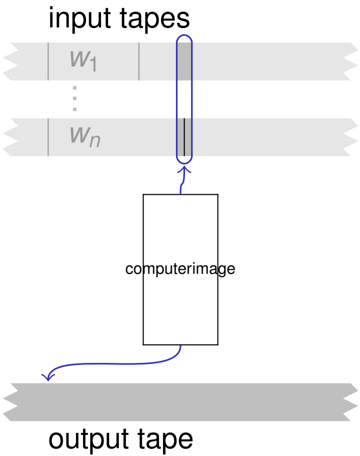

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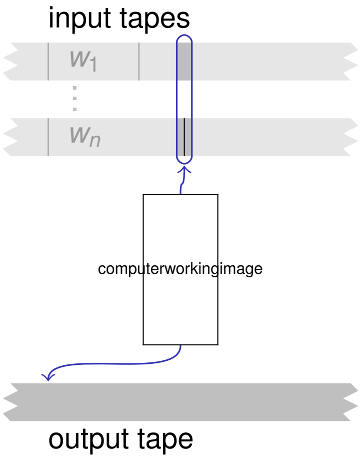

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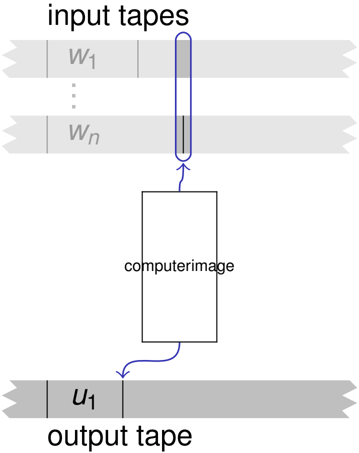

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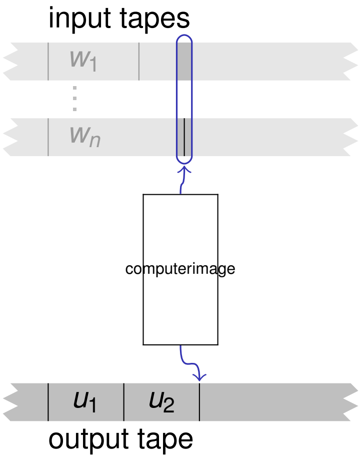

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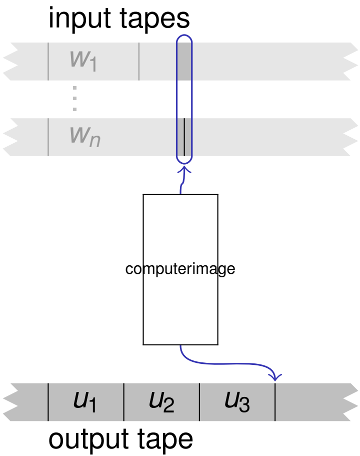

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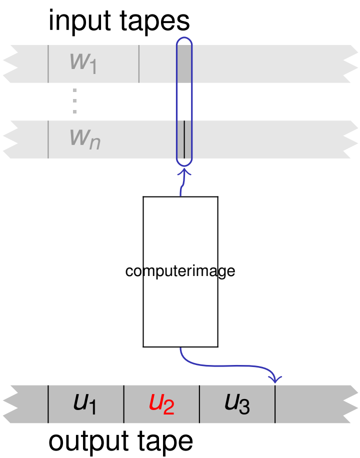

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How Well Can the Cardinality Function Be Enumerated?

Observation

For fixed n, the cardinality function $\#_A^n$

- can be 1-enumerated by Turing machines only for recursive A, but
- can be $(n+1)$-enumerated for every language A.

Question

What about 2-, 3-, 4-, ..., n-enumerability?

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How Well Can the Cardinality Function Be Enumerated by Turing Machines?

Cardinality Theorem (Kummer, 1992)

If $\#_A^n$ is n-enumerable by a Turing machine, then A is recursive.

Weak Cardinality Theorems (1987, 1989, 1992)

1. *If χ_A^n is n-enumerable by a Turing machine, then A is recursive.*
2. *If $\#_A^2$ is 2-enumerable by a Turing machine, then A is recursive.*
3. *If $\#_A^n$ is n-enumerable by a Turing machine that never enumerates both 0 and n, then A is recursive.*

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How Well Can the Cardinality Function Be Enumerated by Finite Automata?

Conjecture

If $\#_A^n$ is n-enumerable by a finite automaton, then A is regular.

Weak Cardinality Theorems (2001, 2002)

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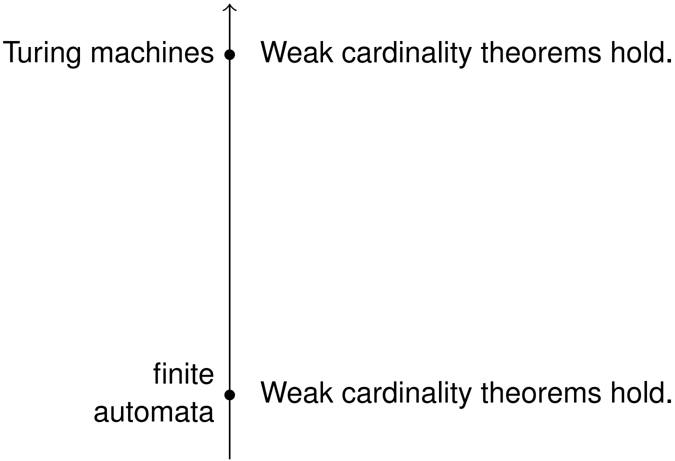
Cardinality Theorems Do Not Hold for All Models



Cardinality Theorems Do Not Hold for All Models

Turing machines	Weak cardinality theorems hold.
resource-bounded machines	Weak cardinality theorems do not hold.
finite automata	Weak cardinality theorems hold.

Why?

First Explanation

The weak cardinality theorems hold both for recursion and automata theory by coincidence.

Second Explanation

The weak cardinality theorems hold both for recursion and automata theory, because they are instantiations of single, unifying theorems.

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The weak cardinality theorems hold both for recursion and automata theory by coincidence.

Second Explanation

The weak cardinality theorems hold both for recursion and automata theory, because they are instantiations of single, unifying theorems.

The second explanation is correct.
The theorems can (almost) be unified using first-order logic.

Outline

What Are Elementary Definitions?

Definition

A relation R is elementarily definable in a logical structure $\mathcal{S}$ if

1. there exists a first-order formula ϕ,
2. that is true exactly for the elements of R.

Example

The set of even numbers is elementarily definable in $(\mathbb{N}, +)$ via the formula $\phi(x) \equiv \exists z \,.\, z + z = x$.

Example

The set of powers of 2 is not elementarily definable in $(\mathbb{N}, +)$.

Characterisation of Classes by Elementary Definitions

Theorem (Büchi, 1960)

There exists a logical structure $(\mathbb{N}, +, \mathrm{e}_2)$ such that a set $A \subseteq \mathbb{N}$ is regular iff it is elementarily definable in $(\mathbb{N}, +, \mathrm{e}_2)$.

Theorem

There exists a logical structure $\mathcal{R}$ such that a set $A \subseteq \mathbb{N}$ is recursively enumerable iff it is positively elementarily definable in $\mathcal{R}$.

Characterisation of Classes by Elementary Definitions

recursively enumerable sets — positively in $\mathcal{R}$

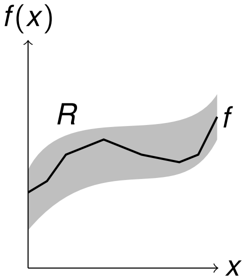
resource-bounded classes — none

regular sets — $(\mathbb{N}, +, e_2)$

Characterisation of Classes by Elementary Definitions

ordinal number arithmetic	$(\mathrm{On}, +, \cdot)$
arithmetic hierarchy	$(\mathbb{N}, +, \cdot)$
recursively enumerable sets	positively in $\mathcal{R}$
resource-bounded classes	none
regular sets	$(\mathbb{N}, +, \mathrm{e}_2)$
Presburger arithmetic	$(\mathbb{N}, +)$

Elementary Enumerability is a Generalisation of Elementary Definability



Definition

A function f is
elementarily m-enumerable in a structure $\mathcal{S}$
if

1. its graph is contained in an elementarily definable relation R,
2. which is m-bounded, i.e., for each x there are at most m different y with $(x, y) \in R$.

The Original Notions of Enumerability are Instantiations

Theorem

A function is m-enumerable by a finite automaton iff it is elementarily m-enumerable in $(\mathbb{N}, +, \mathrm{e}_2)$.

Theorem

A function is m-enumerable by a Turing machine iff it is positively elementarily m-enumerable in $\mathcal{R}$.

The First Weak Cardinality Theorem

Theorem

Let $\mathcal{S}$ be a logical structure with universe U and let $A \subseteq U$. If

1. *$\mathcal{S}$ is well-orderable and*
2. *χ_A^n is elementarily n-enumerable in $\mathcal{S}$,*

then A is elementarily definable in $\mathcal{S}$.

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Corollary

If χ_A^n is n-enumerable by a finite automaton, then A is regular.

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Corollary (with more effort)

If χ_A^n is n-enumerable by a Turing machine, then A is recursive.

The Second Weak Cardinality Theorem

Theorem

Let $\mathcal{S}$ be a logical structure with universe U and let $A \subseteq U$. If

1. *$\mathcal{S}$ is well-orderable,*
2. *every finite relation on U is elementarily definable in $\mathcal{S}$, and*
3. *$\#_A^2$ is elementarily 2-enumerable in $\mathcal{S}$,*

then A is elementarily definable in $\mathcal{S}$.

The Third Weak Cardinality Theorem

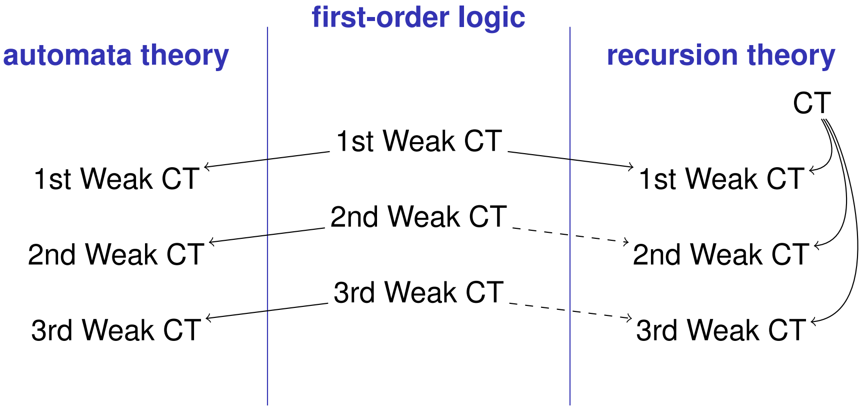
Theorem

Let $\mathcal{S}$ be a logical structure with universe U and let $A \subseteq U$. If

1. *$\mathcal{S}$ is well-orderable,*
2. *every finite relation on U is elementarily definable in $\mathcal{S}$, and*
3. *$\#_A^n$ is elementarily n-enumerable in $\mathcal{S}$ via a relation that never 'enumerates' both 0 and n,*

then A is elementarily definable in $\mathcal{S}$.

Relationships Between Cardinality Theorems (CT)



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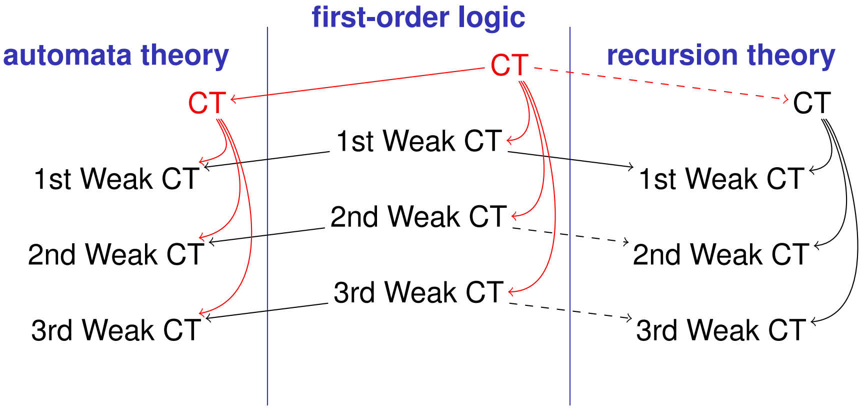

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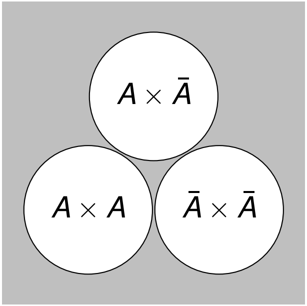


Theorem

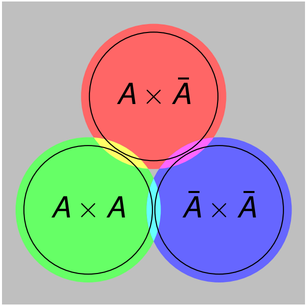
Let $\mathcal{S}$ be a well-orderable logical structure in which all finite relations are elementarily definable.

If there exist elementarily definable supersets of $A \times A$, $A \times \bar{A}$, and $\bar{A} \times \bar{A}$ whose intersection is empty, then A is elementarily definable in $\mathcal{S}$.

Note

The theorem is no longer true if we add $\bar{A} \times A$ to the list.

A Separability Result for First-Order Logic



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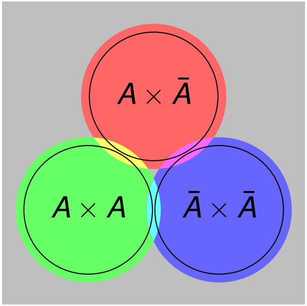

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Summary

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- The weak cardinality theorems for first-order logic unify the weak cardinality theorems of automata and recursion theory.
- The logical approach yields weak cardinality theorems for other computational models.
- Cardinality theorems are separability theorems in disguise.

Open Problems

- Does a cardinality theorem for first-order logic hold?
- What about non-well-orderable structures like $(\mathbb{R}, +, \cdot)$?