Series 9

Exercise 1. Let f, g be two continuous functions and $X_0 \in \mathrm{L}^2(\Omega)$. Solve

$$\begin{aligned} &\mathrm{d}X(t) = f(t)X(t)\,\mathrm{d}t + g(t)X(t)\,\mathrm{d}W(t) \quad t \in [0,T],\\ &X(0) = X_0. \end{aligned}$$

Exercise 2. Let g be a continuous function $b > 0$ and $X_0 \in \mathrm{L}^2(\Omega)$. Solve

$$\begin{aligned} &\mathrm{d}X(t) = -bX(t)\,\mathrm{d}t + g(t)\,\mathrm{d}W(t) \quad t \in [0,T],\\ &X(0) = X_0. \end{aligned}$$

Exercise 3. Let $(X(t), t \geq 0)$ be an n-dimensional process that has the stochastic differential

$$\mathrm{d}X(t) = F(t)\,\mathrm{d}t + G(t)\,\mathrm{d}W(t),$$

where $F(t) \in \mathbb{R}^n$, $G(t) \in \mathbb{R}^{n\times m}$ and $W(t)$ is an m-dimensional Brownian motion. Consider $u(x) : \mathbb{R}^n \to \mathbb{R}$, given by $u(x) = |x|^2 = \sum_{i=1}^n x_i^2$. Compute the stochastic differential $\mathrm{d}u(X(t))$ and give its expression in integral form.

Exercise 4. Let $c, \alpha_1, \ldots, \alpha_m$ be real numbers and $W(t)$ an m-dimensional Brownian motion. Show that the stochastic process $X(t) = e^{ct+\sum_{j=1}^m \alpha_j W_j(t)}$ $t \geq 0$ is solution of the stochastic differential equation

$$\begin{aligned} &\mathrm{d}X(t) = \Big(c + \tfrac{1}{2}\textstyle\sum_{j=1}^m \alpha_j^2\Big)X(t)\,\mathrm{d}t + X(t)\Big(\textstyle\sum_{j=1}^m \alpha_j\,\mathrm{d}W_j(t)\Big) \quad t \in [0,T],\\ &X(0) = 1. \end{aligned}$$

Exercise 5. Assume that $F(t,x) : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function with continuous partial derivatives $\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}, \frac{\partial^2 F}{\partial x^2}$. Assume further that $\frac{\partial F}{\partial x}(t,x) = f(t,x)$. Show the following analogue of the fundamental theorem of Leibniz–Newton calculus for the Itô calculus,

$$\int_a^b f(t, W(t))\,\mathrm{d}W(t) = F(t, W(t))\Big|_a^b - \int_a^b \frac{\partial F}{\partial t}(t, W(t)) + \frac{1}{2}\frac{\partial f}{\partial x}(t, W(t)\,\mathrm{d}t.$$

In particular, when $F(t,x) = F(x)$ is independent of t,

$$\int_a^b f(W(t))\,\mathrm{d}W(t) = F(W(t))\Big|_a^b - \frac{1}{2}\int_a^b f'(W(t))\,\mathrm{d}t.$$

Use this formula to compute $\displaystyle\int_0^t W(s)e^W(s)\,\mathrm{d}W(s)$.

Exercise 6. Let $b > 0$, $\sigma \in \mathbb{R}$ and $X_0 \in \mathbb{R}$ and consider the Langevin equation (X is also called the Ornstein–Uhlenbeck process)

$$\begin{aligned} &\mathrm{d}X(t) = -bX(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t) \quad t \in [0,T],\\ &X(0) = X_0. \end{aligned} \tag{1}$$

i) Solve the equation (1).

ii) Verify that $\lim_{t\to\infty} \mathbb{E}(X(t)) = 0$ and $\lim_{t\to\infty} \mathrm{Var}(X(t)) = \frac{\sigma^2}{2b}$ and that the limiting process is $N\Big(0, \frac{\sigma^2}{2b}\Big)$.

Remark. Note that the limiting process is still $N\Big(0, \frac{\sigma^2}{2b}\Big)$ if X_0 is a Gaussian random variable independent of the Brownian motion.

The stochastic θ method applied to the SDE

$$\begin{aligned} &\mathrm{d}X(t) = f(t, X(t))\,\mathrm{d}t + g(t, X(t))\,\mathrm{d}W(t) \quad t \in [0, T], \\ &X(0) = X_0 \end{aligned}$$

is defined for $0 \leq \theta \leq 1$ and a partition $P = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ as

$$\begin{aligned} X_{n+1} = X_n + f(t_n, X_n)\Delta t(1-\theta) + f(t_{n+1}, X_{n+1})\Delta t\theta + g(t_n, X_n)(W(t_{n+1}) - W(t_n)), \\ \Delta t = t_{n+1} - t_n, \quad n = 0, \ldots, N-1. \end{aligned}$$

For a given $\varepsilon > 0$, we set $\sigma = \sqrt{2/\varepsilon}$, $b = 1/\varepsilon$, $X_0 = 1$ and $T = 1$ and apply the θ method to approximate the solution $X(t)$ of (1) with $\theta = 0, 1/2, 1$.

iii) For $\varepsilon = 1/20, 1/40, 1/80$ and $\theta = 0, 1/2, 1$ what is the largest step size allowed in the θ method ?

iv) For $\varepsilon = 1/20, 1/40, 1/80$ and $\theta = 0, 1/2, 1$, approximate the distribution density of X_{N_k} on uniform partitions $P = \{0 = t_0 < t_1 < \ldots < t_{N_k} = 1\}$ with $N_k = 2^k$, $k = 7, \ldots, 0$. Do the methods conserve the distribution density of the limiting process $N\Big(0, \frac{\sigma^2}{2b}\Big) = N(0, 1)$?
Hint : To approximate the density function f of X_{N_k}, make a histogram of X_{N_k} for $M = 50000$ sample paths and normalize it so that $\int_{\mathbb{R}} f\,\mathrm{d}x = 1$. See the function `hist` in Matlab and take for example `nbins=100`.
