Series 8

Exercise 1. Write the integral expression corresponding to the stochastic differential for $u(t, X_1(t), \ldots, X_n(t))$, where $X_i(t)$ are Itô processes given by $\mathrm{d}X_i = F_i\,\mathrm{d}t + \sum_{j=1}^m G_{ij}\,\mathrm{d}W_j$.

Exercise 2. Let $(W_1(t), t \geq 0)$ and $(W_2(t), t \geq 0)$ be independent one-dimensional standard Brownian motions. Without using the multi-dimensional Itô formula, show that

$$\mathrm{d}(W_1 W_2) = W_1\,\mathrm{d}W_2 + W_2\,\mathrm{d}W_1.$$

Hint : Consider $X(t) = \frac{1}{\sqrt{2}}(W_1(t) + W_2(t))$.

Exercise 3. Let $(W(t), t \geq 0)$ be a one-dimensional standard Brownian motion (for this exercise, you can use the Itô formula).

i) Compute $\mathrm{d}(W^m)$.

ii) Show that $Y(t) = e^{\lambda W(t) - \frac{\lambda^2 t}{2}}$ where $\lambda \in \mathbb{R}$ satisfies

$$\begin{cases} \mathrm{d}Y = \lambda Y\,\mathrm{d}W, \\ Y(0) = 1. \end{cases}$$

Exercise 4. For $n \in \mathbb{N}$, define the n-th Hermite polynomial as

$$h_n(x; \rho) = (-\rho)^n e^{\frac{x^2}{2\rho}} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big(e^{-\frac{x^2}{2\rho}}\Big).$$

i) Compute $h_n(x; \rho)$ for $n = 0, \ldots, 4$.

ii) Show that the Hermite polynomials $\{h_n(x;\rho)\}_{n\geq 0}$ are orthogonal in $\mathrm{L}^2(\mathbb{R})$ (independently of ρ) with respect to the weight function $\nu(x) = \frac{1}{\sqrt{2\pi\rho}} e^{-\frac{x^2}{2\rho}}$ and that

$$\int_{\mathbb{R}} \Big(h_n(x;\rho)\Big)^2 \nu(x)\,\mathrm{d}x = n!\rho^n.$$

Hint :

a) Show that

$$e^{tx - \frac{t^2\rho}{2}} = \sum_{n=0}^{\infty} \frac{h_n(x;\rho)}{n!} t^n.$$

b) Show that

$$e^{\rho t s} = \sum_{n,m=0}^{\infty} \frac{t^n s^m}{n! m!} \int_{\mathbb{R}} h_n(x;\rho) h_m(x;\rho) \nu(x)\,\mathrm{d}x.$$

c) Use *b*) to conclude.

Exercise 5. Let $(W(t), t \geq 0)$ be a standard one-dimensional Brownian motion. Show that

$$\int_0^t h_n(W(s); s)\, \mathrm{d}W(s) = \frac{h_{n+1}(W(t); t)}{n+1}. \tag{1}$$

Hint : Use the formula from Exercise 4

$$e^{\lambda x - \frac{\lambda^2 t}{2}} = \sum_{n=0}^{\infty} \frac{h_n(x;t)}{n!} \lambda^n,$$

substitute $x = W(t)$ and use Exercise 3 *ii*).

Remark. From the formula (1), we see that $h_n(W(t); t)$ plays the role that t^n plays in ordinary calculus.