Series 7

Exercise 1. Let $G_m \in \mathrm{M}^2(0,T)$ be a step process. Show that

$$I(t) = \int_0^t G_m(s)\, \mathrm{d}W(s) \quad 0 \leq t \leq T,$$

is a martingale.

Exercise 2. Let $G \in \mathrm{M}^2(0,T)$ with $t \mapsto G(t,\omega)$ continuous for almost every ω. Show that

$$\lim_{\Delta \to 0} \sum_{j=1}^{m} G(t_{j-1})(W(t_j) - W(t_{j-1})) = \int_0^T G(t)\, \mathrm{d}W(t) \quad \text{in } \mathrm{L}^2(\Omega).$$

Exercise 3. Let $(W(t), t \geq 0)$ be a one-dimensional standard Brownian motion. Without using the Itô formula, show that

i) $\mathrm{d}(W^2) = 2W\,\mathrm{d}W + \mathrm{d}t$,

ii) $\mathrm{d}(tW) = W\,\mathrm{d}t + t\,\mathrm{d}W$.

Exercise 4. (Itô product rule) Suppose that for $0 \leq t \leq T$

$$dX_1 = F_1\,\mathrm{d}t + G_1\,\mathrm{d}W,$$
$$dX_2 = F_2\,\mathrm{d}t + G_2\,\mathrm{d}W,$$

where $F_i \in \mathrm{M}^1(0,T)$ and $G_i \in \mathrm{M}^2(0,T)$, $i = 1,2$. Assume that $X_1(0) = X_2(0) = 0$ and $F_i(t) = F_i$ and $G_i(t) = G_i$ are time independent $\mathcal{F}(0)$ measurable random variables $(i = 1,2)$ and show that

$$\mathrm{d}(X_1 X_2) = X_2\,\mathrm{d}X_1 + X_1\,\mathrm{d}X_2 + G_1 G_2\,\mathrm{d}t.$$

Exercise 5. For $\lambda, \mu \in \mathbb{R}$, we consider the SDE

$$\begin{aligned} dX(t) &= \lambda X(t)\,\mathrm{d}t + \mu X(t)\,\mathrm{d}W(t) \quad 0 \leq t \leq T, \\ X(0) &= X_0. \end{aligned} \tag{1}$$

i) Verify that its solution in the Itô sense is given by $X(t) = X_0 e^{(\lambda - \frac{1}{2}\mu^2)t + \mu W(t)}$.

Let $P = \{0 = t_0 < t_1 < \ldots < t_m = T\}$ be a partition of $[0,T]$ of size Δt and define the Euler polygonal interpolant of W on P as

$$\hat{W}(t) = W(t_{n-1}) + (W(t_n) - W(t_{n-1}))\frac{t - t_{n-1}}{t_n - t_{n-1}} \quad t_{n-1} \leq t \leq t_n \;\; 1 \leq i \leq N.$$

ii) If we replace W with $\hat{W}$ in the SDE (1), we obtain the ordinary differential equation

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{X}^m(t) = \lambda \hat{X}^m(t) + \mu \hat{X}^m(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{W}(t) \quad 0 \le t \le T, \\ &\hat{X}^m(0) = X_0. \end{aligned} \tag{2}$$

Give the solution $\hat{X}^m(t)$ of (2).

iii) What is the limit in $\mathrm{L}^2(\Omega)$ of $\hat{X}^m(t)$ as $m \to \infty$ ($\Delta \to 0$) ?

In order to approximate numerically the solution of (2), we use (2) to obtain

$$\hat{X}^m(t_{n+1}) = \hat{X}^m(t_n) + \lambda \hat{X}^m(t_n)\Delta t + \mu(W(t_{n+1}) - W(t_n))\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \hat{X}^m(s)\,\mathrm{d}s, \tag{3}$$

where $\Delta t = t_{n+1} - t_n$. To construct a numerical method for the approximation of $\hat{X}^m$, we need to approximate the integral $\int_{t_n}^{t_{n+1}} \hat{X}^m(s)\,\mathrm{d}s$.

iv) If we approximate the integral with the Euler formula $\int_{t_n}^{t_{n+1}} \hat{X}^m(s)\,\mathrm{d}s \approx \Delta t \hat{X}^m(t_n)$ what method do we obtain ? What does this method converge to ? What is the strong order of convergence of the method ?

v) We now approximate the integral with the trapezoidal rule

$$\int_{t_n}^{t_{n+1}} \hat{X}^m(s)\,\mathrm{d}s \approx \frac{\Delta t}{2}(\hat{X}^m(t_n) + \hat{X}^m(t_{n+1})),$$

and make an Euler prediction for the implicit term

$$\hat{X}^m(t_{n+1}) \approx \hat{X}^m(t_n) + \lambda \hat{X}^m(t_n)\Delta t + \mu \hat{X}^m(t_n)(W(t_{n+1}) - W(t_n)).$$

Write the obtained method for a uniform partition of $[0, 1]$.

vi) Let $\lambda = 2$, $\mu = 1$, $X_0 = 1$ and $T = 1$ and consider a uniform partition of $[0, 1]$. Write a Matlab code to compute the approximation given by the method of *v*). What solution does this method converge to ? What is the strong order of convergence of the method ?

General information and series on `http://anmc.epfl.ch/num_int_SDE.html`