Series 6

Exercise 1. Let $(W(t), t \geq 0)$ be a one-dimensional Brownian motion. Show that

$$P(W(t) \in B | W(s)) = \frac{1}{\sqrt{2\pi(t-s)}} \int_B e^{-\frac{|x-W(s)|^2}{2(t-s)}} \, \mathrm{d}x \quad \text{a.s.}$$

for all $0 \leq s < t$ and all Borel set $B \in \mathcal{B}$.

Exercise 2. Let

$$I(P, \lambda) = \sum_{j=1}^{m} W(\tau_j)(W(t_j) - W(t_{j-1})),$$

where $P = \{0 = t_0 < t_1 < \ldots < t_m = t\}$ is a partition of $[0, t]$ of size Δ and $\tau_j = (1-\lambda)t_{j-1} + \lambda t_j$, $0 \leq \lambda \leq 1$ is an intermediate point. Define $I_\lambda(t) = \frac{W(t)^2}{2} + (\lambda - \frac{1}{2})t$ the L^2 limit of $I(P, \lambda)$ as $\Delta \to 0$. Show that $I_\lambda(t)$ is a martingale if and only if $\lambda = 0$.

Exercise 3. (Brownian bridge) Consider the interval $[0, 1]$. A Brownian bridge is a standard Gaussian process $(Z(t), 0 \leq t \leq 1)$ such that $\mathrm{Cov}(Z(t), Z(s)) = \min\{s, t\} - st$. Let $(W(t), 0 \leq t \leq 1)$ be a standard Brownian motion.

i) Show that $Z(t) = W(t) - tW(1)$ is a Brownian bridge.

In some applications, we would like to construct a modified Wiener process $(Z(t), 0 \leq t \leq T)$ for which all sample paths satisfy $Z(0) = x$, $Z(T) = y$.

ii) Using the Brownian bridge, construct such a process that is Gaussian with $\mathbb{E}(Z(t)) = x - \frac{t}{T}(x - y)$ and $\mathrm{Cov}(Z(t), Z(s)) = \min\{s, t\} - \frac{st}{T}$.

iii) Write a Matlab code to simulate a Brownian bridge $(Z(t), 0 \leq t \leq 2)$ with $Z(0) = 1$, $Z(2) = 2$ on a partition with $\Delta t = 2^{-4}, 2^{-6}, 2^{-8}$. Approximate $\mathbb{E}(Z(t))$ over $M = 20, 200, 2000$ trajectories.

Exercise 4. (Step 2 in the construction of Itô integral) Let $T > 0$ and $G \in \mathrm{M}^2(0, T)$ be bounded i.e. $|G(t, \omega)| \leq M \quad \forall (t, \omega)$.

i) Construct an explicit continuous positive function $\psi_m(x) : \mathbb{R} \to \mathbb{R}$ such that $\psi_m(x) = 0$ for $x \in]-\infty, -\frac{1}{m}] \cup [0, \infty[$ and $\int_{\mathbb{R}} \psi_m(x) \, \mathrm{d}x = 1$.

ii) Show that for all $\omega \in \Omega$

$$t \mapsto G_m(t, \omega) = \int_0^T \psi_m(s - t) G(s, \omega) \, \mathrm{d}s,$$

is continuous on $[0, T]$.

iii) Show that

$$\mathbb{E}\Big(\int_0^T (G(t,\omega) - G_m(t,\omega))^2 \,\mathrm{d}t\Big) \to 0 \text{ as } m \to \infty.$$

Hint : use the Lebesgue differentiation theorem : Let $f : \mathbb{R}^d \to \mathbb{R}$ an integrable function. Then, for almost every $a \in \mathbb{R}^d$,

$$\lim_{r\to 0} \frac{1}{B_r(a)} \int_{B_r(a)} f(x)\,\mathrm{d}x = f(a),$$

where $B_r(a) = \{x \in \mathbb{R}^d : |x-a| < r\}$.

Exercise 5. (Paley–Wiener–Zygmund stochastic integral) The purpose of this exercise is to construct a stochastic integral $\int_0^T g(s)\,\mathrm{d}W(s)$ for deterministic function $g : [0,T] \to \mathbb{R}$.

i) Denote by $\mathcal{S}$ the set of step functions of the form $g(s) = \sum_{i=1}^m a_i \chi_{[t_{i-1},t_i[}(s)$ for some $0 = t_0 < t_1 < \ldots < t_m = T$ and $a_i \in \mathbb{R}$. For $g \in \mathcal{S}$, define

$$\int_0^T g(s)\,\mathrm{d}W(s) = \sum_{i=1}^m a_i(W(t_i) - W(t_{i-1})).$$

Compute the mean and the second moment of $\int_0^T g(s)\,\mathrm{d}W(s)$.

ii) Show that the application $g(s) \mapsto \int_0^T g(s)\,\mathrm{d}W(s)$ is an isometry from $\mathcal{S} \subset \mathrm{L}^2(0,T)$ to $\mathrm{L}^2(\Omega,\mathcal{F},P)$ that can be uniquely extended to $\mathrm{L}^2(0,T)$. This isometry is called the Paley–Wiener–Zygmund integral.

iii) Show that for $f \in \mathrm{L}^2(0,T)$, $\int_0^T f(s)\,\mathrm{d}W(s) \sim N(0, \int_0^T f(s)^2\,\mathrm{d}s)$.

iv) Show that for $g : [0,T] \to \mathbb{R}$ continuously differentiable with $g(0) = g(T) = 0$, the Paley–Wiener–Zygmund integral can be alternatively defined as

$$\int_0^T g(s)\,\mathrm{d}W(s) = -\int_0^T g'(s)W(s)\,\mathrm{d}s.$$

Hint : Check that $\mathbb{E}(\int_0^T g(s)\,\mathrm{d}W(s)) = 0$ and $\mathbb{E}\Big(\Big(\int_0^T g(s)\,\mathrm{d}W(s)\Big)^2\Big) = \int_0^T g(s)^2\,\mathrm{d}s$.