Series 5

Exercise 1. Let $(W(t), t \geq 0)$ be a one-dimensional Brownian motion. For $0 \leq t_1 < \ldots < t_n$, show that the joint density $f_X(x)$, where $X = (W(t_1), \ldots, W(t_n))$, is given by

$$f_X(x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} e^{-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}},$$

where $x = (x_1, \ldots, x_n)$ and $t_0 = 0, x_0 = 0$.

Exercise 2. The family of Haar functions $\{h_k\}_{k\geq 0}$ is defined for $0 \leq t \leq 1$ as follows :

$$h_0(t) = 1 \quad \text{for } 0 \leq t \leq 1,$$

$$h_1(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1/2 \\ -1 & \text{for } 1/2 < t \leq 1, \end{cases}$$

and for $2^n \leq k < 2^{n+1}$, $n = 1, 2, \ldots$ we set

$$h_k(t) = \begin{cases} 2^{n/2} & \text{for } \frac{k-2^n}{2^n} \leq t \leq \frac{k-2^n+1/2}{2^n} \\ -2^{n/2} & \text{for } \frac{k-2^n+1/2}{2^n} < t < \frac{k-2^n+1}{2^n} \\ 0 & \text{otherwise.} \end{cases}$$

The Schauder functions $\{s_k\}_{k\geq 0}$ are then defined as $s_k(t) = \langle \chi_{[0,t]}, h_k \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathrm{L}^2(0,1)$.

i) Show that $\{h_k\}_{k\geq 0}$ is orthonormal in $\mathrm{L}^2(0,1)$.

ii) Show that $\{h_k\}_{k\geq 0}$ is complete in $\mathrm{L}^2(0,1)$, i.e. $f = \sum_{k=0}^{\infty} \langle f, h_k \rangle h_k$ in $\mathrm{L}^2(0,1)$ for any $f \in \mathrm{L}^2(0,1)$.
Hint : first show that if $\langle f, h_k \rangle = 0$ for all $k \geq 0$ then $f = 0$ a.s.

iii) Show that $\sum_{k=0}^{\infty} s_k(s) s_k(t) = \min\{s, t\}$ for any $0 \leq s, t \leq 1$

Exercise 3. We define $W(t) = \sum_{k=0}^{\infty} s_k(t)\xi_k$, where $\{s_k\}_{k=0}^{\infty}$ are the Schauder functions introduced in Exercise 2 and $\{\xi_k\}_{k\geq 0}$ is a sequence of independent $N(0,1)$ random variables.

i) Show that there exists a constant C such that for sufficiently large $k \geq 2$ we have $|\xi_k| \leq C\sqrt{\log k}$ a.s.
Hint : find a bound for $P(|\xi_k| \geq 4\sqrt{\log k})$ and apply Borel–Cantelli lemma.

ii) Show that the series $W(t)$ converges uniformly for $t \in [0,1]$.

Exercise 4. For a given $n \geq 1$, we approximate a Brownian motion as

$$W_n(t) = \sum_{k=0}^{2^{n+1}-1} s_k(t)\xi_k,$$

where $\{s_k\}_{k=0}^{2^{n+1}-1}$ are the Schauder functions introduced in Exercise 2 and $\{\xi_k\}_{k=0}^{2^{n+1}-1}$ is a sequence of independent $N(0,1)$ random variables. Let $P = \{0 = t_0 < t_1 < \ldots < t_N = 1\}$ be the uniform partition of $[0,1]$ with $\Delta t = 2^{-12}$.

i) For $n = 3, \ldots, 10$, compute W_n on the partition P.

ii) Verify numerically that $\mathbb{E}(W_n(t)) = 0$.

iii) Observe numerically that the sequence $V_n = \sum_{i=1}^{N} |W_n(t_i) - W_n(t_{i-1})|$ diverges.

iv) Consider the series of the time derivative of W

$$D_n(t) = \tfrac{\mathrm{d}}{\mathrm{d}t} W_n(t) = \sum_{k=0}^{2^{n+1}-1} h_k(t)\xi_k.$$

For $n = 3, \ldots, 10$ compute D_n on the partition P and observe that the series diverges.

Exercise 5. Consider the following Riemann sum,

$$I(P, \lambda) = \sum_{j=1}^{m} W(\tau_j)(W(t_j) - W(t_{j-1})),$$

for a given uniform partition $P = \{0 = t_0 < t_1 < \ldots < t_m = T\}$ with size $\Delta = t_j - t_{j-1}$ and $\tau_j = (1-\lambda)t_{j-1} + \lambda t_j$, $0 \leq \lambda \leq 1$. For given λ, T, approximate the following limit in $\mathrm{L}^2(\Omega)$ (the method to approximate W can be chosen)

$$L(\lambda, T) = \lim_{\Delta \to 0} I(P, \lambda) - \frac{W(T)^2}{2} \quad \text{in } \mathrm{L}^2(\Omega).$$

i) Does the limit $L(\lambda, T)$ converge ? (try for $\lambda = 0, 1/2, 1$ and $T = 1, 2$)

ii) How does the limit $L(\lambda, T)$ depend on λ and T ?
For $(T, \lambda) \in \{1, 2, 3\} \times \{0, 1/4, 1/2, 3/4, 1\}$ approximate the limit (with a constant $\Delta = 2^{-10}$ and for example $M = 50000$ sample paths).

General information and series on `http://anmc.epfl.ch/num_int_SDE.html`