Series 4

Exercise 1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G} \subseteq \mathcal{F}$ a σ-algebra. Let $X, Y : \Omega \to \mathbb{R}$ be integrable random variables on Ω. Prove that

i) $\mathbb{E}(aX + bY|\mathcal{G}) = a\mathbb{E}(X|\mathcal{G}) + b\mathbb{E}(Y|\mathcal{G})$ a.s. for $a, b \in \mathbb{R}$.

ii) If X is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$ a.s.

iii) If X is $\mathcal{G}$-measurable and XY is integrable, then $\mathbb{E}(XY|\mathcal{G}) = X\mathbb{E}(Y|\mathcal{G})$ a.s.

iv) If X is independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ a.s.

v) If $\mathcal{H}$ is a σ-algebra with $\mathcal{H} \subseteq \mathcal{G}$, then

$$\mathbb{E}(X|\mathcal{H}) = \mathbb{E}\Big(\mathbb{E}(X|\mathcal{G})\Big|\mathcal{H}\Big) = \mathbb{E}\Big(\mathbb{E}(X|\mathcal{H})\Big|\mathcal{G}\Big) \text{ a.s.}$$

vi) If $X \leq Y$, then $\mathbb{E}(X|\mathcal{G}) \leq \mathbb{E}(Y|\mathcal{G})$ a.s.

Exercise 2. (Conditional Jensen inequality) Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is Borel measurable and convex with $\mathbb{E}|\phi(X)| < \infty$, where $X : \Omega \to \mathbb{R}$ is a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. Show that

$$\phi\Big(\mathbb{E}(X|\mathcal{G})\Big) \leq \mathbb{E}(\phi(X)|\mathcal{G}),$$

where $\mathcal{G} \subseteq \mathcal{F}$ is a σ-algebra.
Hint : We recall that for a convex function ϕ and $x_0 \in \mathbb{R}$, there exists a number $\lambda(x_0)$ such that

$$\phi(x) \geq \phi(x_0) + \lambda(x_0)(x - x_0) \quad \forall x \in \mathbb{R}. \tag{1}$$

Exercise 3. Let $B(t)$ be a one-dimensional standard Gaussian process and assume $\mathbb{E}(B(t)B(s)) = \min\{s, t\}$. Show that $B(t)$ is a Brownian motion.
We recall that a real valued stochastic process $(X(t), t \geq 0)$ is called a one-dimensional Gaussian process if for any $n \geq 1$ and any choice of times $t_1, \ldots, t_n$, the random vector $(X(t_1), \ldots, X(t_n))$ has a multivariate Gaussian distribution. It is called a standard one-dimensional Gaussian process if $E(X(t)) = 0$ for all $t \geq 0$.

Exercise 4. Suppose that $(W(t), t \geq 0)$ is a one-dimensional standard Brownian motion. Show that

i) $X(t) = \frac{1}{a}W(a^2t)$ is a standard Brownian motion for $a \neq 0$.

ii) $X(t) = \begin{cases} tW\left(\frac{1}{t}\right) & t > 0 \\ 0 & t = 0 \end{cases}$ is a standard Brownian motion.

iii) $X(t) = W(t + t_0) - W(t_0)$ is a standard Brownian motion for $t_0 \geq 0$.

iv) $X(t) = W(T-t) - W(T)$, where $t \in [0,T]$ and $T > 0$ is a standard Brownian motion.

Exercise 5. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\{X_n\}_{n\geq 1}$ a sequence of real valued random variables $X_n : \Omega \to \mathbb{R}$ and $X : \Omega \to \mathbb{R}$ another random variable. We denote by F_n and F the distribution function of X_n and X, respectively.

- We say that $X_n \to X$ in L^2 if $\mathbb{E}(|X_n - X|^2) \to 0$, as $n \to \infty$.
- We say that $X_n \to X$ in probability if $\forall \varepsilon > 0$ $P(|X_n - X| > \varepsilon) \to 0$ as $n \to \infty$.
- We say that $X_n \to X$ in distribution if $\lim_{n\to\infty} F_n(x) = F(x)$ for all x for which F is continuous.

i) Prove that $X_n \to X$ in L^2 implies $X_n \to X$ in probability.

ii) Prove that $X_n \to X$ in probability implies $X_n \to X$ in distribution.

iii) Show with counter examples that $X_n \to X$ in distribution does not imply $X_n \to X$ in probability and $X_n \to X$ in probability does not imply $X_n \to X$ in L^2.
Hint : For $\varepsilon > 0$, show that for all $n \geq 1$,

$$F(x-\varepsilon) - P(|X_n - X| > \varepsilon) \leq F_n(x) \leq F(x+\varepsilon) + P(|X_n - X| > \varepsilon).$$

Show that the limit in $\mathrm{L}^2(\Omega)$ of a convergent sequence of Gaussian random variables is a Gaussian random variable.