Series 3

Exercise 1. Consider an SDE

$$\begin{aligned} &dX(t) = f(X(t))dt + g(X(t))dW(t) \quad 0 \leq t \leq T, \\ &X(0) = X_0. \end{aligned} \tag{1}$$

and suppose we want to approximate $\mathbb{E}(\phi(X(T))) =: Z$. Denote X_N an approximation of $X(T)$ obtained with an Euler–Maruyama method with time step $\Delta t = T/N$ and assume that we have a random sample of X_N $(X_N^1, \dots, X_N^M)$. To approximate the expectation Z we compute $\hat{Z} = \frac{1}{M}\sum_{i=1}^M \phi(X_N^i)$. To measure the accuracy of this estimator we consider the root mean square error defined by

$$\sqrt{\text{MSE}(\hat{Z})} = \sqrt{\mathbb{E}((Z - \hat{Z})^2)}.$$

i) Show that we have the decomposition

$$\text{MSE}(\hat{Z}) = \text{Var}(\hat{Z}) + (\text{bias}(\hat{Z}))^2.$$

where $\text{Var}(\hat{Z})$ is the variance of $\hat{Z}$ and $\text{bias}(\hat{Z}) = \mathbb{E}(\hat{Z}) - Z$.

ii) Give an expression of $\text{Var}(\hat{Z})$ in term of $\text{Var}(\phi(X_N))$ and an upper bound of $\text{bias}(\hat{Z})$ so that

$$\text{MSE}(\hat{Z}) = \mathcal{O}\Big(\tfrac{1}{M} + (\Delta t)^2\Big). \tag{2}$$

Exercise 2.

i) We define the computational cost of the Monte-Carlo estimator $\hat{Z}$ of Exercise 1 as

$$\mathcal{O}(\text{number of time steps} \cdot \text{number of path simulation}).$$

Assume that we want an accuracy such that $\sqrt{\text{MSE}(\hat{Z})} = \mathcal{O}(\varepsilon)$. Compute the computational cost needed to obtain this accuracy.

ii) Suppose that we have a method of weak order $p \geq 1$, what is the computational cost to obtain an estimator $\hat{Z}$ with root mean square error $\mathcal{O}(\varepsilon)$.

iii) Minimize MSE (as a function of M and Δt) subject to a fixed computational cost $\eta = M/\Delta t$. Show that for the optimal $M, \Delta t$, we have $\sqrt{\text{MSE}(\hat{Z})} = \mathcal{O}\Big(\eta^{-\frac{p}{1+2p}}\Big)$.

Exercise 3. Consider The EM method of Series 1 and the following scheme (called Milstein scheme),

$$\begin{aligned} X_n = X_{n-1} + f(X_{n-1})\Delta t + g(X_{n-1})(W(t_n) - W(t_{n-1})) \\ + \tfrac{1}{2} g'(X_{n-1}) g(X_{n-1}) \Big((W(t_n) - W(t_{n-1}))^2 - \Delta t \Big). \end{aligned}$$

Apply the Milstein method to the SDE of Series 1 :

$$\begin{aligned} &dX(t) = \lambda X(t)dt + \mu X(t)dW(t) \quad 0 \leq t \leq T, \\ &X(0) = X_0. \end{aligned} \tag{3}$$

where $\lambda = 2$, $\mu = 1$. Establish experimentally the strong and weak order of convergence.

Exercise 4. For (3), we want to observe numerically that estimate (2) holds. We set $\phi = \mathrm{id}$ and approximate $Z = \mathbb{E}(X(T)) = e^{\lambda T}$ using the EM scheme and the Monte-Carlo method. The purpose is to observe when the step size is small, the Monte-Carlo error $\mathcal{O}(\frac{1}{M})$ becomes dominant in (2). To do so, compute $M = 100, 1000, 10000$ sample paths X_{N_p} on grids of size $\Delta t_p = 2^p \delta t$, $p = 7, \ldots, 0$, $\delta t = 2^{-10}$ and compute the estimator $\hat{Z}_p$. To compute $\mathbb{E}((Z - \hat{Z}_p)^2)$, use a Monte-Carlo method with $S = 50$ samplings, i.e.,

$$\mathrm{MSE}(\hat{Z}_p) = \mathbb{E}((Z - \hat{Z}_p)^2) \approx \frac{1}{S} \sum_{s=1}^{S} (Z - \hat{Z}_{p,s})^2,$$

For each M, plot in loglog axes the values $(\Delta t_p, \mathrm{MSE}(\hat{Z}_p))$.