Series 2

This series of exercises is a "revision" on standard notions in probability theory. Students having difficulties in solving these problems should take advantage to revise the related probability theory background (see the references in the course page).

Exercise 1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{F_n\}_{n\geq 1}$ a sequence of $\mathcal{F}$. Show that

i) if $F_n \subseteq F_{n+1}$ $\forall n \geq 1$ then $P\Big(\bigcup_{n=1}^{\infty} F_n\Big) = \lim_{n\to\infty} P(F_n)$.

ii) if $F_n \supseteq F_{n+1}$ $\forall n \geq 1$ then $P\Big(\bigcap_{n=1}^{\infty} F_n\Big) = \lim_{n\to\infty} P(F_n)$.

Let $\{F_n\}_{n\geq 1}$ a sequence of $\mathcal{F}$. Show that there exists a sequence $\{F'_n\}_{n\geq 1}$ with $F'_i \cap F'_j = \emptyset$ if $i \neq j$ and

$$\bigcup_{n=1}^{\infty} F'_n = \bigcup_{n=1}^{\infty} F_n.$$

Exercise 2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{A_i\}_{i\geq 1}$ a sequence of events $A_i \in \mathcal{F}$.

i) Show that the event $E = \{\text{infinitely many } A_i \text{ occurs}\}$ can be written

$$E = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i.$$

Remark. The set E is often called "A_i i.o." , which means "A_i infinitely often".

ii) Describe the event $H = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i$.

iii) (Borel–Cantelli) Show that if $\sum_{i=1}^{\infty} P(A_i) < \infty$ then $P(A_i \text{ i.o.}) = 0$.

Exercise 3. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}^n$ an n-dimensional random variable. Show that $\mathcal{F}(X) = \{X^{-1}(B); B \in \mathcal{B}\}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^n$, is a σ-algebra. Observe that $\mathcal{F}(X)$ is the smallest σ-algebra with respect to which X is measurable.

Exercise 4. Let $(\Omega, \mathcal{F}, P)$ be a probability space and consider $X, Y : \Omega \to \mathbb{R}$ two random variables. Provide an example to illustrate that $X \overset{\text{d}}{=} Y$ and $X \overset{\text{a.s.}}{=} Y$ are two different equality concepts.

Exercise 5. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Consider $X : \Omega \to \mathbb{R}^n$ an n-dimensional random variable and assume $X \in \mathrm{L}^p(\Omega)$ $1 \leq p \leq \infty$.

i) Show that $\|X\|_q \leq \|X\|_p$ for any q with $1 \leq q \leq p$.

ii) For $p = 2$, $\mathrm{L}^2(\Omega)$ is a Hilbert space. Give the corresponding scalar product. For real valued random variables $X, Y : \Omega \to \mathbb{R}$, is $\mathrm{Cov}(X,Y) = \mathbb{E}\Big((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\Big)$ a scalar product ?

iii) (Markov's inequality) Show that for any $\varepsilon > 0$

$$P(|X| \geq \varepsilon) \leq \tfrac{1}{\varepsilon^p}\mathbb{E}(|X|^p).$$

Deduce that

$$P(|X - \mathbb{E}(X)| \geq \varepsilon) \leq \frac{\mathrm{Var}(X)}{\varepsilon^2}.$$

iv) Assume that a numerical method for a SDE has strong order of convergence 1/2, i.e. $\mathbb{E}|X_n - X(t_n)| \leq C(\Delta t)^{1/2}$ (for example the EM method in Series 1). Show that

$$P\Big(|X_n - X(t_n)| < (\Delta t)^{1/4}\Big) \geq 1 - C(\Delta t)^{1/4}.$$

Remark. This shows that the strong convergence of the method implies a bound on the error of individual simulation (pointwise error).

Exercise 6. Let $(\Omega, \mathcal{F}, P)$ be a probability space and consider $X : \Omega \to \mathbb{R}^n$ a random variable with density $f_X : \mathbb{R}^n \to \mathbb{R}$. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective function such that $g^{-1} = h$ is continuously differentiable and $\det J_h \neq 0$, where J_h is the Jacobian matrix of h. Show that $Y = g(X)$ is a random variable with density $f_Y(y) = f_X(h(y))|\det J_h|$.

Exercise 7. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X : \Omega \to \mathbb{R}^n$ a standard Gaussian random vector with density function $f_X(x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}$, where $x = (x_1, \ldots, x_n)^T$ and $|x| = \Big(\sum_{i=1}^n x_i^2\Big)^{1/2}$.

i) Verify that $\mathbb{E}(X) = 0 \in \mathbb{R}^n$ and $\mathrm{Cov}(X) = I$ ($n \times n$ identity matrix).

ii) Let $Y = AX + b$, where A is an $n \times n$ invertible matrix and $b \in \mathbb{R}^n$. Show that the density function of Y is

$$f_Y(y) = \frac{1}{(2\pi)^{n/2}|\det A|} e^{-\frac{1}{2}(y-b)^T(AA^T)^{-1}(y-b)}.$$

General information and series on `http://anmc.epfl.ch/num_int_SDE.html`