Series 12

Exercise 1. (Weak order for the Milstein–Talay scheme)
Let $f, g : \mathbb{R} \to \mathbb{R}$ fulfill the assumptions on existence and uniqueness of a SDE and consider the following autonomous SDE

$$\begin{aligned} \mathrm{d}X(t) &= f(X(t))\,\mathrm{d}t + g(X(t))\,\mathrm{d}W(t), \\ X(0) &= X_0, \end{aligned}$$

and the scheme

$$\begin{aligned} X_{n+1} = &X_n + f(X_n)h + g(X_n)\Delta W_n + g'(X_n)g(X_n)I_{1,1} \\ &+ \Big(f'(X_n)f(X_n) + \frac{1}{2}f''(X_n)g^2(X_n)\Big)\frac{h^2}{2} + f'(X_n)g(X_n)I_{1,0} \\ &+ \Big(g'(X_n)f(X_n) + \frac{1}{2}g''(X_n)g^2(X_n)\Big)I_{0,1}, \end{aligned} \tag{1}$$

where

$$I_{1,1} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \mathrm{d}W(s_2)\,\mathrm{d}W(s_1), \quad I_{1,0} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \mathrm{d}s_2\,\mathrm{d}W(s_1),$$
$$I_{0,1} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \mathrm{d}W(s_2)\,\mathrm{d}s_1.$$

Remark. Under additional assumptions on f and g, it can be shown using the theorem on weak convergence of the lecture that the above scheme has weak order 2.

i) Derivative free scheme : Let $\{\xi_n\}$ be a sequence of independent discrete random variables, ξ_n being independent of X_n and such that

$$P(\xi_n = \pm\sqrt{3}) = \frac{1}{6}, \qquad P(\xi_n = 0) = \frac{2}{3},$$

Noting $\chi_n = \xi_n h^{1/2}$, the derivative scheme is defined as

$$\begin{aligned} Z_1 &= X_n + f(X_n)h + g(X_n)\chi_n, \\ Z_2^{\pm} &= X_n + f(X_n)h \pm g(X_n)\sqrt{h}, \\ X_{n+1} &= X_n + \frac{1}{2}\Big(f(Z_1) + f(X_n)\Big)h + \frac{1}{4}\Big(g(Z_2^+) + g(Z_2^-) + 2g(X_n)\Big)\chi_n \\ &\quad + \frac{1}{4}\Big(g(Z_2^+) - g(Z_2^-)\Big)\Big(\chi_n^2 - h\Big)\frac{1}{\sqrt{h}}. \end{aligned} \tag{2}$$

Show that the following derivative free scheme (2) is consistent with (1), i.e. by Taylor expansion we recover (1) with possibly higher order terms and the random variables having the same moments properties.

ii) Verify numerically that the schemes (1) and (2) have weak order 2 : Consider the SDE

$$\begin{aligned} dX(t) &= \lambda X(t)dt + \mu X(t)dW(t) \quad 0 \le t \le T, \\ X(0) &= X_0. \end{aligned} \tag{3}$$

where $\lambda = 2$, $\mu = 0.1$, $X_0 = 1$, $T = 1$. Note that (1) can be applied to the linear SDE (3) as we can show that $I_{1,0} + I_{0,1} = h\Delta W_n$.

Exercise 2. Consider the solution X of the SDE

$$\begin{aligned} \mathrm{d}X(t) &= f(t, X(t))\,\mathrm{d}t + g(t, X(t))\,\mathrm{d}W(t) \quad 0 \le t \le T, \\ X(0) &= X_0, \end{aligned}$$

and $\{X_n\}_{n=0}^N$ its approximation obtained with the Euler–Maruyama method. Consider the multilevel Monte–Carlo method (MLMC) introduced in the course. Recall that

$$\bar{Z} = \sum_{\ell=0}^{L} \frac{1}{M_\ell} \sum_{i=1}^{M_\ell} \phi_\ell^{(i)} - \phi_{\ell-1}^{(i)},$$

where we use the same Brownian path for $\phi_\ell^{(i)}$ and $\phi_{\ell-1}^{(i)}$ but independent Brownian paths for $\phi_\ell^{(i)} - \phi_{\ell-1}^{(i)}$ and $\phi_\ell^{(j)} - \phi_{\ell-1}^{(j)}$ $i \neq j$. We showed in the course that

$$\mathrm{Var}(\bar{Z}) = \sum_{\ell=0}^{L} \frac{V_\ell}{M_\ell}.$$

where $V_\ell = \mathrm{Var}(\phi_\ell - \phi_{\ell-1})$. Assume that ϕ is Lipschitz continuous, i.e. $|\phi(x) - \phi(y)| \le K|x - y|$ for some constant K and show that

$$V_\ell \le C h_\ell,$$

for a constant C independent of h_ℓ.

Exercise 3. (facultative) Implement the MLMC method for the Euler–Maruyama scheme. Consider the SDE

$$\begin{aligned} \mathrm{d}X(t) &= \lambda X(t)\,\mathrm{d}t + \mu X(t)\,\mathrm{d}W(t) \quad 0 \le t \le T, \\ X(0) &= X_0, \end{aligned}$$

with $\lambda = 1$, $\mu = 0.1$, $T = 1$, $X_0 = 0.1$ and consider the two functionals $\phi(x) = x$ and $\phi(x) = x^2$.

i) Compute the decay of the variance $\mathrm{Var}(\phi_\ell - \phi_{\ell-1})$ as a function of the level ℓ. What happens for $\mathrm{Var}(\phi_\ell)$?

ii) Compare the computational cost versus error for both the MLMC method and the standard Monte–Carlo method (set on the x axis the computational budget and monitor the corresponding MSE).

General information and series on `http://anmc.epfl.ch/num_int_SDE.html`