Series 11

Exercise 1. Show by considering the following simple SDE

$$\begin{aligned} \mathrm{d}X(t) &= \lambda X(t)\,\mathrm{d}t + \mu X(t)\,\mathrm{d}W(t), \\ X(0) &= X_0, \end{aligned}$$

where $\{W(t), t \geq 0\}$ is a one-dimensional standard Brownian motion that the fully implicit method

$$X_{n+1} = X_n + \lambda X_{n+1} h + \mu X_{n+1} \Delta W_n,$$

has unbounded first moments, i.e., $\mathbb{E}|X_n| = \infty$.

Exercise 2.

i) Let $f, g_1, \ldots, g_m : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $W_1(t), \ldots, W_m(t)$ be m independent one-dimensional Brownian motions, and consider the SDE

$$\begin{aligned} \mathrm{d}X(t) &= f(t, X(t))\,\mathrm{d}t + \sum_{j=1}^{m} g_j(t, X(t))\,\mathrm{d}W_j(t), \\ X(0) &= X_0, \end{aligned} \tag{1}$$

By following the same procedure as in the 1d case, we obtain the Milstein–Platten scheme $\{X_n\}_{n \geq 0}$ whose k-th component is

$$\begin{aligned} Z_j &= X_n + f(t_n, X_n)h + g_j(t_n, X_n)\sqrt{h}, \quad j = 1, \ldots, m, \\ X_{n+1}^k &= X_n^k + f^k(t_n, X_n)h + \sum_{j=1}^{m} g_j^k(t_n, X_n)\Delta_n W_j + \frac{1}{\sqrt{h}} \sum_{ij=1}^{m} \Big(g_i^k(t_n, Z_j) - g_i^k(t_n, X_n) \Big) I_{i,j}, \end{aligned}$$

where $I_{i,j} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} \mathrm{d}W_i(s_2)\,\mathrm{d}W_j(s_1)$, for $i, j \in \{1, \ldots, m\}$. What is the issue for $m > 1$?

ii) Consider (1) with $m = d$ and "diagonal noise", i.e., $g_j(t,x) = (g_j^1, \ldots, g_j^d)$ are such that $g_j^k = 0$ for $k \neq j$ and $\partial_{x_j} g_k^k = 0$ for $k \neq j$. Show that the multi-dimensional Milstein–Platten scheme can be written in that case as (for the k-th component)

$$\begin{aligned} Z_k &= X_n + f(t_n, X_n)h + g_k(t_n, X_n)\sqrt{h}, \\ X_{n+1}^k &= X_n^k + f^k(t_n, X_n)h + g_k^k(t_n, X_n)\Delta_n W_k \\ &\quad + \frac{1}{2} g_k^k(t_n, X_n)\partial_{x_k} g_k^k(t_n, X_n)\Big((\Delta_n W_k)^2 - h \Big). \end{aligned}$$

Exercise 3. Let $f(t,x), g(t,x) : [0,T] \times \mathbb{R} \to \mathbb{R}$ fulfill the assumption on existence and uniqueness of an SDE. Verify that The Euler–Maruyama method satisfies the hypotheses of Lemma 1, i.e.

i) $\mathbb{E}(X_{n+1} - X_n | X_n = x) \leq C(1 + |x|)h$, *ii)* $|X_{n+1} - X_n| \leq M_n(1+|X_n|)\sqrt{h}$, where M_n is a random variable independent of X_n such that $\mathbb{E}|M_n|^r < C_r$ where C_r is a constant independent of n, h..

Exercise 4. (Weak order for the Euler–Maruyama method)
Let $f(t,x), g(t,x) : [0,T] \times \mathbb{R} \to \mathbb{R}$ fulfill the assumption on existence and uniqueness of an SDE. Assume further that $x \mapsto f(t,x)$ and $x \mapsto g(t,x)$ are in $\mathcal{C}_b^2$ uniformly in $[0,T]$, $x \mapsto \partial_t f(t,x), \partial_t g(t,x) \in \mathcal{C}_b^0$ uniformly in $[0,T]$ and $\varphi \in \mathcal{C}_p^4$. Let $X(t)$ be the solution of the SDE

$$\begin{aligned} \mathrm{d}X(t) &= f(t, X(t))\,\mathrm{d}t + g(t, X(t))\,\mathrm{d}W(t), \\ X(0) &= X_0, \end{aligned}$$

and $\{X_n\}_{n\geq 0}$ be the approximation of X obtained with the Euler–Maruyama method.

i) Show that

$$\mathbb{E}\Big(\varphi(X(h))\Big|X(0) = x\Big) = \varphi(x) + h\mathcal{L}\varphi(x) + R_1,$$

where $\mathcal{L}\varphi(x) = f(t,x)\partial_x\varphi(x) + \frac{1}{2}g^2(t,x)\partial_{xx}\varphi(x)$ and $|\mathbb{E}(R_1)| \leq C(1+\mathbb{E}|x|^\kappa)h^2$, for some $C > 0$ and $\kappa \in \mathbb{N}$.
Hint : Use Itô formula.

ii) Show that

$$\mathbb{E}\Big(\varphi(X_1)\Big|X(0) = x\Big) = \varphi(x) + h\mathcal{L}\varphi(x) + R_2,$$

where $\mathcal{L}\varphi(x) = f(t,x)\partial_x\varphi(x) + \frac{1}{2}g^2(t,x)\partial_{xx}\varphi(x)$ and $|\mathbb{E}(R_2)| \leq C(1+\mathbb{E}|x|^\kappa)h^2$, for some $C > 0$ and $\kappa \in \mathbb{N}$.
Hint : Denote $X_1 = X_0 + \Delta X$, define $F(t) = \varphi(X_0 + t\Delta X)$ and Taylor expand $F(1)$.

iii) Conclude from *i)* and *ii)* that the Euler–Maruyama method has weak order 1.

Exercise 5. Consider the modified Euler–Maruyama method given by

$$X_{n+1} = X_n + f(t_n, X_n)h + g(t_n, X_n)\xi_n,$$

where $\{\xi_n\}$ is a sequence of independent random variables such that ξ_n is independent of X_n and

$$\mathbb{E}(\xi_n^\ell) = 0,\ \ell = 1,3, \quad \mathbb{E}(\xi_n^2) = h, \quad |\mathbb{E}(\xi_n^4)| < \infty.$$

i) Prove that this method has weak order 1.

ii) Verify numerically on the SDE that this method has weak order 1 (proceed similarly as in Exercise 4 of Series 1).