Series 10

Exercise 1. Let $\{X_n\}_{n\geq 1}$ be a sequence of real valued random variables. We say that $X_n \to X$ a.s. if $\forall \varepsilon > 0$ $P(\lim_{n\to\infty} |X_n - X| \leq \varepsilon) = 1$.

i) Show that if $X_n \to X$ in $\mathrm{L}^2(\Omega)$, there exists a subsequence $\{n_k\}_{k\geq 1}$ such that $X_{n_k} \to X$ a.s.

ii) Give the details of the Step 3 in the proof of the existence and uniqueness of a solution of an SDE and show that

$$X(t) = X_0 + \int_0^t f(s, X(s))\,\mathrm{d}s + \int_0^t g(s, X(s))\,\mathrm{d}W(s) \quad \text{a.s. } \forall t \in [0, T],$$

and also that

$$P\Big(\omega \in \Omega : X(t) = X_0 + \int_0^t f(s, X(s))\,\mathrm{d}s + \int_0^t g(s, X(s))\,\mathrm{d}W(s) \quad \forall t \in [0, T]\Big) = 1.$$

Exercise 2. Under the assumptions of the theorem on existence and uniqueness of a solution of an SDE, show that

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |X(t)|^2\Big) \leq C\Big(1 + \mathbb{E}|X_0|^2\Big).$$

Hint : Follow the idea of the beginning of Step 2.

Exercise 3. Let $W_1, \ldots, W_m$ be m independent one-dimensional Brownian motions. Consider

$$I_{i_1,\ldots,i_p} = \int_0^h \int_0^{s_1} \cdots \int_0^{s_{p-1}} \mathrm{d}W_{i_p}(s_p) \ldots \mathrm{d}W_{i_1}(s_1),$$

where

$$\mathrm{d}W_{i_j}(s_j) = \begin{cases} \mathrm{d}s_j & \text{if } i_j = 0, \\ \mathrm{d}W_{i_j}(s_j) & \text{if } i_j = j \in \{1, \ldots, p\}. \end{cases}$$

For example $I_{0,2,1} = \int_0^h \int_0^{s_1} \int_0^{s_2} \mathrm{d}W_1(s_3)\,\mathrm{d}W_2(s_2)\,\mathrm{d}s_1$. Show that

i) $\mathbb{E}(I_{i_1,\ldots,i_p}) = 0$ if at least one $i_j \neq 0$.
ii) $\mathbb{E}(I_{i_1,\ldots,i_p}) = \mathcal{O}(h^p)$ if all $i_j = 0$, $j = 1, \ldots, p$.
iii) $\mathbb{E}(I^2_{i_1,\ldots,i_p})^{1/2} = \mathcal{O}(h^q)$, where $q = \sum_{j=1}^p \frac{2-\min\{1,i_j\}}{2}$.

In particular, observe that $\mathbb{E}|I_{i_1,\ldots,i_p}| = \mathcal{O}(h^q)$.

Exercise 4. Consider the following SDE

$$\begin{aligned} &\mathrm{d}X(t) = \lambda X(t)\,\mathrm{d}t + \mu X(t)\,\mathrm{d}W(t) \quad t \geq 0, \\ &X(0) = X_0, \end{aligned} \tag{1}$$

where $\lambda, \mu \in \mathbb{R}$. We say that a solution of this SDE is mean-square stable if

$$\lim_{t\to\infty} \mathbb{E}(|X(t)|^2) = 0.$$

If $\mathbb{E}|X_0|^2 \neq 0$, show that

$$\lim_{t\to\infty} \mathbb{E}|X(t)|^2 = 0 \quad \Leftrightarrow \quad (\lambda,\mu) \in S = \{(\lambda,\mu) \in \mathbb{R}^2 : \lambda + \frac{1}{2}\mu^2 < 0\}.$$

S is called the stability region of the SDE (1). For $(\lambda, \mu) \in \mathbb{R}^2$, plot the stability region of (1) on the (x, y)-axis choosing $x = \lambda$, $y = \mu^2$.

Exercise 5. Let $\{X_n\}_{n\geq 0}$ be the approximation of (1) obtained with the θ method with time step h (see Series 9). For given h, θ, we say that the sequence $\{X_n\}_{n\geq 1}$ is mean-square stable if

$$\lim_{n\to\infty} \mathbb{E}(|X_n|^2) = 0.$$

We define the stability domain

$$S_\theta = \{(p,q) \in \mathbb{R}^2 : p = h\lambda, q = \sqrt{h}\mu, \ \lim_{n\to\infty} \mathbb{E}(|X_n|^2) = 0\}.$$

i) Compute S_θ and plot it for $\theta = 0, 1/2, 1$ on the (x, y)-axis choosing $x = h\lambda$, $y = h\mu^2$.

ii) Show that

$$\begin{array}{ll} S_\theta \subset S & \text{for } 0 \leq \theta < \frac{1}{2}, \\ S_\theta = S & \text{for } \theta = \frac{1}{2}, \\ S_\theta \supset S & \text{for } \frac{1}{2} < \theta \leq 1. \end{array}$$

iii) Let $\lambda = -3$, $\mu = \sqrt{3}$ and check numerically that for $\theta \geq \frac{1}{2}$, $\lim_{n\to\infty} |X_n| = 0$. Why ?
Hint : approximate X up to $T = 20$ is sufficient in this example to observe the asymptotic behaviour.

iv) For $\theta < \frac{1}{2}$, give a sufficient condition on h such that $\lim_{n\to\infty} |X_n| = 0$. Is this condition necessary ?
Hint : approximate X up to $T = 20$ with $h = 1, 1/2, 1/4$.
