Series 1

Exercise 1. Consider a scalar standard Brownian motion (Wiener process) on $[0, 1]$.

i) Write a Matlab code to simulate a discretized Brownian motion on $t_j = j\Delta t$ with $\Delta t = 2^{-4}, 2^{-6}, 2^{-8}$.

ii) Compute the mean over 20, 200, 2000 trajectories.

iii) Compute the discretized stochastic process $X(t) = X_0 e^{(\lambda - \frac{1}{2}\mu^2)t + \mu W(t)}$ on $t_j = j\Delta t$, for $\lambda = 2$, $\mu = 1$, $X_0 = 1$ with $\Delta t = 2^{-4}, 2^{-6}, 2^{-8}$.

iv) Compute the mean of $X(t)$ over 20, 200, 2000 trajectories. Can you guess what $\mathbb{E}(X(t))$ is?

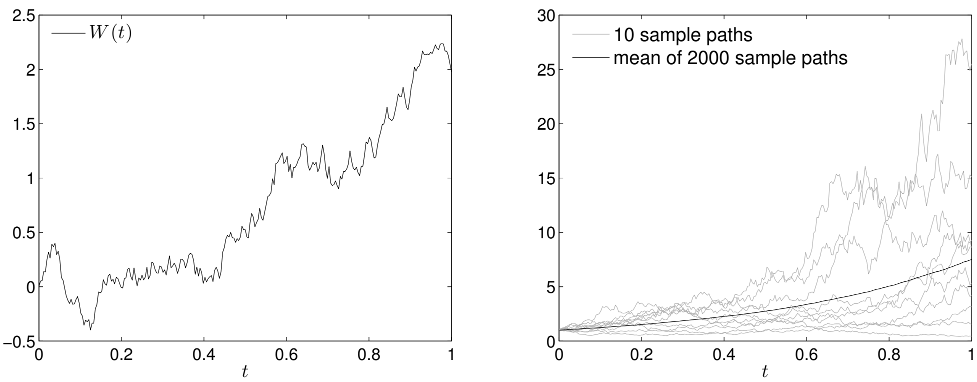

Figure 1: Left : A discretized Brownian motion for $\Delta t = 2^{-8}$. Right : 10 sample paths of the process $e^{(\lambda - \frac{1}{2}\mu^2)t + \mu W(t)}$ and mean over 2000 sample paths ($\Delta t = 2^{-8}$, `rng(2015)`).

Exercise 2. Let $\lambda = 2$, $\mu = 1$ and consider the stochastic differential equation

$$\begin{aligned} \mathrm{d}X(t) &= \lambda X(t)\,\mathrm{d}t + \mu X(t)\,\mathrm{d}W(t) \quad 0 \le t \le T, \\ X(0) &= X_0. \end{aligned} \tag{1}$$

and the Euler–Maruyama (EM) method for (1)

$$X_n = X_{n-1} + \lambda X_{n-1}\Delta t + \mu X_{n-1}(W(t_n) - W(t_{n-1})).$$

The exact solution of (1) is given by (see Exercise 1) $X(t) = X_0 e^{(\lambda - \frac{1}{2}\mu^2)t + \mu W(t)}$. Compute a discretized Brownian path over $[0, 1]$ with $\delta t = 2^{-8}$ and compare the exact solution (on the discretized path) with the EM method (using the same Brownian path) with $\Delta t = 2^4\delta t, 2^2\delta t$.

Exercise 3. (Strong convergence)
A numerical method for an SDE

$$\begin{aligned} \mathrm{d}X(t) &= f(X(t))\,\mathrm{d}t + g(X(t))\,\mathrm{d}W(t) \quad 0 \le t \le T, \\ X(0) &= X_0. \end{aligned} \tag{2}$$

is said to have a *strong order of convergence equals to* r if there exists a constant C such that

$$\mathbb{E}|X_n - X(t_n)| \leq C(\Delta t)^r,$$

for any fixed $t_n = n\Delta t \in [0, T]$. For the SDE of Exercise 2 set $t_n = T = 1$, apply the EM method and set $e^{\mathrm{s}}_{\Delta t} := \mathbb{E}|X_n - X(t_n)|$. Verify numerically that $e^{\mathrm{s}}_{\Delta t} \leq C(\Delta t)^{1/2}$. To evaluate $\mathbb{E}|X_n - X(t_n)|$ you need to compute $\frac{1}{M}\sum_{i=1}^{M} |X^i_n - X^i(t_n)|$, i.e. the average over M realizations of the random variables at time $t_n = 1$. For that :

i) take $M = 10000$ independent discretized Brownian path over $[0, 1]$ with $\delta t = 2^{-9}$;
ii) for each path apply EM with $\Delta t = 2^p \delta t$, $0 \leq p \leq 4$ and store the endpoint error (at $t = T$) ;
iii) take the mean over the error and report the result (Δt versus strong error) in a loglog plot.

Exercise 4. (Weak convergence)
A numerical method for an SDE

$$\begin{aligned} &\mathrm{d}X(t) = f(X(t))\,\mathrm{d}t + g(X(t))\,\mathrm{d}W(t) \quad 0 \leq t \leq T \\ &X(0) = X_0. \end{aligned} \tag{3}$$

is said to have a *weak order of convergence equals to* r if there exists a constant C such that

$$|\mathbb{E}p(X_n) - \mathbb{E}p(X(t_n))| \leq C(\Delta t)^r,$$

for any fixed $t_n = n\Delta t \in [0, T]$ and all sufficiently smooth function p. For the SDE of Exercise 2 with $\lambda = 2$, $\mu = 0.1$, set $t_n = T = 1$ and $e^{\mathrm{w}}_{\Delta t} := |\mathbb{E}(X_n) - \mathbb{E}(X(t_n))|$ and verify numerically that $e^{\mathrm{w}}_{\Delta t} \leq C\Delta t$.

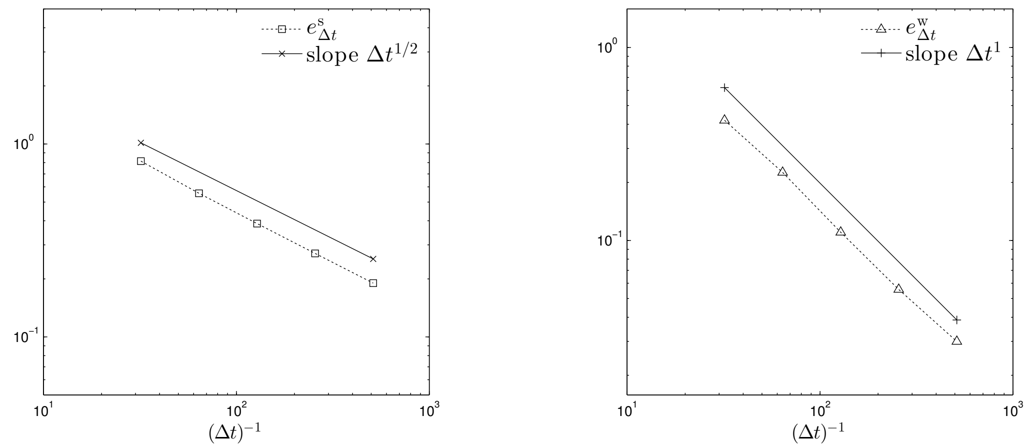

Figure 2: Left : Strong order of convergence 1/2 for the EM method (Exercise 3). Right : Weak order of convergence 1 for the EM method (Exercise 4).

General information and series on `http://anmc.epfl.ch/num_int_SDE.html`