Series 9

Exercise 1. Show by a direct computation that the so-called "symplectic Euler method" is symplectic.

Exercise 2. Consider a general Hamiltonian system of the form

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}(p,q), \quad \dot{q}_i = \frac{\partial H}{\partial p_i}(p,q), \qquad i = 1,\dots,d, \tag{1}$$

where $p(t), q(t) \in \mathbb{R}^d$ and $H(p,q)$ denotes the Hamiltonian.

The aim is to prove the following statement (E. Celledoni, R. McLachlan, D. McLaren, B. Owren, G. Quispel and W. Wright, *Energy-preserving Runge–Kutta methods*, ESAIM: M2AN, vol. 43, 2009.)

No (consistent) Runge–Kutta method exactly preserves the Hamiltonian for all polynomial Hamiltonian systems. However, for any given polynomial Hamiltonian, there exists a (consistent) Runge–Kutta method that exactly preserves it.

i) Show that a (consistent) Runge–Kutta method cannot exactly preserve the Hamiltonian $H(p,q)$ for all Hamiltonian systems (1) with $H(p,q)$ a polynomial.
Hint: For $d=1$, consider the Hamiltonian $H(p,q) = p - \int_0^q g(t)\,\mathrm{d}t$ where $g(t)$ is some polynomial.

ii) Consider the "average vector field method" (AVF)

$$y_{n+1} = y_n + h\int_0^1 f(\theta y_{n+1} + (1-\theta)y_n)\,\mathrm{d}\theta.$$

Assume that the vector field $f(y)$ is Lipschitz continuous. Show that this method is well-defined for a stepsize h small enough.

iii) Show that the AVF method exactly conserves the energy $H(y)$ for any system (1).
Hint: $H(y_{n+1}) - H(y_n) = \int_0^1 \frac{d}{d\theta}\Big(H(\theta y_{n+1} + (1-\theta)y_n)\Big)\,\mathrm{d}\theta.$

iv) Assume that the Hamiltonian function $H(p,q)$ is polynomial. Show that there exists a quadrature formula $(b_i, c_i)_{i=1,\dots,s}$, with nodes c_i and weights b_i, such that

$$\int_0^1 f(\theta y_{n+1} + (1-\theta)y_n)\,\mathrm{d}\theta = \sum_{i=1}^s b_i f(y_n + (y_{n+1}-y_n)c_i),$$

where $f(y) = J^{-1}\nabla H(y)$. Then, construct a Runge–Kutta method that exactly preserves this particular polynomial Hamiltonian $H(p,q)$.

Exercise 3. Find a consistent, irreducible Runge–Kutta method that is symplectic, but different from any Gauss method.
Hint: Construct a two-stage method satisfying the conditions of the Cooper theorem.

General information and series on `http://anmc.epfl.ch/Numerical.html`