Series 8

Exercise 1. Consider the system of differential equations

$$\dot{y} = f(y), \qquad y(0) = y_0,$$

and let $\varphi_h(y_0)$ denote the associated flow at time $t = h$. Let Φ_h be a one step method of order p satisfying

$$\Phi_h(y_0) = \varphi_h(y_0) + C(y_0)h^{p+1} + \mathcal{O}(h^{p+2}).$$

i) Show that the adjoint method Φ_h^* satisfies

$$\Phi_h^*(y_0) = \varphi_h(y_0) + (-1)^p C(y_0)h^{p+1} + \mathcal{O}(h^{p+2}).$$

ii) Show that the (maximal) order of a symmetric method is even.

Exercise 2.

i) Show that the adjoint of a collocation method based on the nodes $c_1, c_2, \ldots, c_s$ is again a collocation method based on the nodes $c_1^*, c_2^*, \ldots, c_s^*$, where $c_i^* = 1 - c_{s+1-i}$, for $i = 1, 2, \ldots, s$.

ii) Deduce from *i)* that the Gauss collocation methods are symmetric.

Exercise 3. Consider the Hamiltonian system

$$\dot{y} = J^{-1}\nabla H(y) = f(y), \qquad y(0) = y_0, \tag{1}$$

and its variational equation

$$\dot{\Psi} = \frac{\partial f}{\partial y}\Psi, \qquad \Psi(0) = I. \tag{2}$$

i) Show that trace $\left(\frac{\partial f}{\partial y}\right) = 0$ and hence that $\det \Psi$ is a first integral of (2).

ii) Show that for every bounded open set $\Omega \subset \mathbb{R}^{2d}$ and for every $t \in \mathbb{R}$, for which the flow φ_t of (1) exists, we have (Liouville's theorem)

$$\mathrm{Vol}(\varphi_t(\Omega)) = \mathrm{Vol}(\Omega),$$

where $\mathrm{Vol}(\Omega) = \int_\Omega dy$.

iii) More generally, show that the flow of a general system of differential equations $\dot{y} = f(y)$ in $\mathbb{R}^n$ is volume preserving if and only if $\operatorname{div} f(y) = 0$ for all $y \in \mathbb{R}^n$.

Exercise 4.

i) Prove that a smooth transformation $g : \mathbb{R}^2 \to \mathbb{R}^2$ is symplectic if and only if it is volume and orientation preserving.

ii) Is this statement still true for a transformation $g : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ with $d > 1$? If not, find a counterexample.