Series 7

Exercise 1. (Isospectral flows)
We consider the differential equation

$$\dot{L} = [B(L), L], \qquad L(0) = L_0, \tag{1}$$

where L_0 is a $d \times d$ symmetric matrix, $B(L)$ is a $d \times d$ skew-symmetric matrix for all L and $[B, L] = BL - LB$, the Lie-bracket.
Show that the solution $L(t)$ of (1) is a symmetric matrix for any t and that its eigenvalues are independent of t (hence the name "isospectral flow").
Hint: Consider the differential equation $\dot{U} = B(UL_0U^T)U$, $U(0) = I$ and show that the solution for (1) is given by $L(t) = U(t)L_0U(t)^T$. Then use Exercise 2 of Series 5.

Exercise 2. (Isospectral methods)
We consider the differential equation (1), where L_0 is a $d \times d$ symmetric matrix, $B(L)$ is a $d \times d$ skew-symmetric matrix for all L and $[B, L] = BL - LB$, the Lie-bracket. Exercise 1 shows that the characteristic polynomial $\det(L(t) - \lambda I) = \sum_{i=0}^{d} \alpha_i(t)\lambda^i$ has coefficients $\alpha_i(t)$ independent of t. Thus, these coefficients are polynomial invariants as $\alpha_0 = \det L$, $\alpha_{d-1} = (-1)^{d-1}\operatorname{trace} L$, etc.

We know from the lecture that it is not possible for a Runge–Kutta method to conserve all polynomials of degree d for $d \geq 3$. Hence, in general, a Runge–Kutta method cannot conserve the quantities $\alpha_0, \ldots, \alpha_d$. Thus, a Runge–Kutta method cannot be isospectral for all problems of the form (1).

However, an idea to solve (1) numerically is the following (Calvo, Iserles & Zanna, 1999): Assume that L_n, an approximation to $L(t_n)$, is known. Solve numerically the problem

$$\dot{U} = B(UL_nU^T)U, \qquad U(0) = I, \tag{2}$$

from $t = 0$ to $t = h$ and denote its solution by U_1^n. Then set $L_{n+1} = U_1^n L_n (U_1^n)^T$ for obtaining an approximation of $L(t_{n+1})$.
Question: Find a numerical method to solve (1) which ensures that L_{n+1} and L_n have the same eigenvalues.

Exercise 3. Consider the differential equation

$$y' = f(y), \qquad y(0) = y_0.$$

We assume that f is locally Lipschitz continuous and for simplicity, we assume that the solution exists for all $t \in \mathbb{R}$. Consider the flow of the differential equation

$$\varphi_t(y_0) = y(t, 0, y_0),$$

denoting the solution at time t with initial value equal to y_0 at time 0.

i) Show that $\varphi_t \circ \varphi_s(y_0) = \varphi_{t+s}(y_0)$ and hence $\varphi_t \circ \varphi_{-t} = \mathrm{Id}$, for $t, s \in \mathbb{R}$.
ii) Given a numerical method $y_{n+1} = \Phi_h(y_n)$, its adjoint $y_{n+1} = \Phi_h^*(y_n)$ is the method defined as $y_n = \Phi_{-h}(y_{n+1})$ or equivalently $y_{n+1} = \Phi_{-h}^{-1}(y_n)$. We say that a numerical method $y_{n+1} = \Phi_h(y_n)$ is symmetric if it satisfies $\Phi_h \circ \Phi_{-h} = \mathrm{Id}$. Show that

a) Φ_h symmetric $\Leftrightarrow \Phi_h = \Phi_h^*$,
b) $(\Phi_h^*)^* = \Phi_h$,
c) $(\Phi_h \circ \Psi_h)^* = \Psi_h^* \circ \Phi_h^*$ for any one-step methods Φ_h, Ψ_h.

Exercise 4.

i) Are the explicit Euler or the implicit Euler method symmetric methods?
ii) Is the implicit midpoint rule symmetric?
iii) For $y_{n+1} = \Phi_h(y_n)$ a numerical method, show that

$$y_{n+1} = \Phi_{h/2} \circ \Phi_{h/2}^*(y_n), \tag{3}$$

is a symmetric method.
iv) Which methods does (3) yield for the explicit Euler and the implicit Euler method?

Exercise 5. Consider an s-stage Runge–Kutta method that is consistent, i.e., $\sum_{i=1}^s b_i = 1$, and with coefficients such that $\sum_{j=1}^s a_{ij} = c_i$, for $1 \le i \le s$.

i) Show that the adjoint of the Runge–Kutta method is again a Runge–Kutta method, with coefficients given by

$$a_{ij}^* = b_{s+1-j} - a_{s+1-i,s+1-j}, \quad b_i^* = b_{s+1-i} \qquad \text{for } 1 \le i,j \le s.$$

ii) Deduce from *i*) that $a_{ij} = b_j - a_{s+1-i,s+1-j}$ for all $i,j = 1,\dots,s$, if the method is symmetric.
iii) Deduce from *ii*) that, if the Runge–Kutta method is explicit, it cannot be symmetric.

Hints:

a) Use the convention, that we describe a Runge–Kutta method with stages ordered in such a way that $c_1 \le \ldots \le c_s$.
b) Show that the adjoint method can be described by

$$\begin{cases} k_i = f(y_0 + h\sum_{j=1}^s (b_j - a_{ij})k_j), \quad 1 \le i \le s \\ y_1 = y_0 + h\sum_{i=1}^s b_i k_i, \end{cases}$$

and observe that $\sum_{j=1}^s (b_j - a_{ij}) = 1 - c_i$, for $1 \le i \le s$.

General information and series on `http://anmc.epfl.ch/Numerical.html`