Series 6

Exercise 1. (Partitioned Runge–Kutta method) Show that the symplectic Euler method is a first order method and that the Störmer–Verlet method is a second order method.

Exercise 2. Derive the order conditions for the coefficients $\{b_i, a_{ij}\}, \{\hat{b}_i, \hat{a}_{ij}\}$ of a partitioned Runge–Kutta method such that the method is of second order.

Exercise 3. Prove that if a Runge–Kutta method $\{b_i, a_{ij}\}$ satisfies $b_i a_{ij} + b_j a_{ji} = b_i b_j$, for $i, j = 1, \dots, s$ then it conserves all invariants of the form $I(y) = y^T C y + d^T y + c$.

Exercise 4. Show (without using the Cooper theorem) that the symplectic Euler method conserves all quadratic invariants of the form $y^T D z$, with $D \in \mathbb{R}^{n\times m}$, for partitioned systems

$$\dot{y} = f(y, z), \qquad \dot{z} = g(y, z),$$

where $y(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^m$.

Exercise 5. (Cooper theorem for partitioned RK methods) Prove that if the coefficients of a partitioned Runge–Kutta method satisfy

$$\begin{aligned} &b_i \hat{a}_{ij} + \hat{b}_j a_{ji} = b_i \hat{b}_j, \quad i, j = 1, ..., s, \\ &b_i = \hat{b}_i, \quad i = 1, ..., s, \end{aligned}$$

then the method conserves quadratic invariants of the form $Q(y, z) = y^T D z$.

Exercise 6. Let $\{b_i, a_{ij}\}, \{\hat{b}_i, \hat{a}_{ij}\}$ be a partitioned method. Show that if both methods $\{b_i, a_{ij}\}, \{\hat{b}_i, \hat{a}_{ij}\}$ are irreducible and if the partitioned method conserves quadratic invariants of the form

$$y^T C y + 2 y^T D z + z^T E z, \tag{1}$$

with $C \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times m}$, $E \in \mathbb{R}^{m\times m}$, then the Runge–Kutta methods $\{b_i, a_{ij}\}$, $\{\hat{b}_i, \hat{a}_{ij}\}$ define the same numerical scheme.

Hints:

i) If the partitioned method satisfies (1), then each of the two Runge–Kutta methods has to conserve quadratic invariant, respectively.
ii) For irreducible Runge–Kutta methods, the Cooper theorem gives a necessary condition for the conservation of quadratic invariants.
iii) Prove that $b_i a_{ij} = b_i \hat{a}_{ij}$ for all $1 \le i, j \le s$.