Series 5

Exercise 1. (Lotka–Volterra)

i) Show that

$$I(u,v) = u + v - \ln u - 2\ln v,$$

is a first integral of the Lotka–Volterra problem

$$\dot{u} = u(v-2), \qquad \dot{v} = v(1-u). \tag{1}$$

ii) We recall the Cauchy–Lipschitz theorem (enriched with the result about maximality of solutions):
If a vector field f is locally Lipschitz continuous, then for all initial values y_0 the system

$$\dot{y} = f(y), \qquad y(t_0) = y_0,$$

possesses a unique maximal solution $u(t)$ on an interval of the form $[t_0, t_0 + T_{y_0}[$ with length satisfying $0 < T_{y_0} \leq +\infty$.
Let T denote the length of the interval of existence of the solution of (1) for the initial condition

$$u(t_0) = u_0, \qquad v(t_0) = v_0.$$

a) Show that if $u_0 = 0$ then $T = +\infty$ and $u(t) = 0$ for all $t \geq t_0$.
b) Show that if there exists $\tilde{t} \in [t_0, t_0 + T[$ such that $u(\tilde{t}) = 0$ then $u(t) = 0$ for all $t \in [t_0, t_0 + T[$.
c) Show that if $u_0 > 0$ then $u(t) > 0$ for all $t \in [t_0, t_0 + T[$.

Exercise 2. We consider the system

$$\dot{Y} = A(Y)Y, \quad Y(t_0) = Y_0,$$

where Y and $A(Y)$ are $n \times n$ matrices.

i) Show that if $A(Y)$ is a skew-symmetric matrix for all Y then $I(Y) := Y^T Y$ is a first integral of the system.
ii) Show that if in addition Y_0 is orthogonal (i.e., $Y_0^T Y_0 = I$) then the solution $Y(t)$ stays orthogonal for all $t \geq t_0$.

Hint. Notice that *i)* also holds for $\dot{Y} = A(Y)Y$ when Y is a vector which belongs to $\mathbb{R}^n$ and $A(Y)$ is an $n \times n$ skew-symmetric matrix.

Exercise 3. (Euler equations of a rigid body) Consider the system

$$\begin{aligned} \dot{y}_1 &= \alpha_1 y_2 y_3, \qquad & \alpha_1 &= 1/I_3 - 1/I_2, \\ \dot{y}_2 &= \alpha_2 y_3 y_1, \qquad & \alpha_2 &= 1/I_1 - 1/I_3, \\ \dot{y}_3 &= \alpha_3 y_1 y_2, \qquad & \alpha_3 &= 1/I_2 - 1/I_1, \end{aligned}$$

where $I_1, I_2, I_3 > 0$ are constants.

i) Write this problem in the form $\dot{Y} = A(Y)Y$ and find a first integral of the system.

ii) What kind of integrator would you apply to that problem?

iii) Show that another first integral is given by $H(Y) = \frac{1}{2}(\frac{y_1^2}{I_1} + \frac{y_2^2}{I_2} + \frac{y_3^2}{I_3})$.

General information and series on `http://anmc.epfl.ch/Numerical.html`