Series 4

Exercise 1. Recall the definition of a collocation method. Show that if $f(t,y)$ satisfies a Lipschitz condition, a polynomial of degree s satisfying the s (nonlinear) conditions

$$u(t_0) = y_0, \quad u'(t_0 + c_i h) = f(t_0 + c_i h, u(t_0 + c_i h)) \text{ with } i = 1, 2, \ldots, s,$$

does indeed exist for h small enough.
Hint. Use Exercise 5 of Series 2.

Exercise 2. Let $0 \leq c_1 < c_2 < \cdots < c_s \leq 1$ be given. We recall the relations

$$C(q): \qquad \sum_{j=1}^{s} a_{ij} c_j^{k-1} = \frac{c_i^k}{k}, \quad k = 1, \ldots, q,$$

$$B(q): \qquad \sum_{i=1}^{s} b_i c_i^{k-1} = \frac{1}{k}, \qquad k = 1, \ldots, q.$$

Show that the relations $a_{ij} = \int_0^{c_i} \ell_j(\tau) d\tau$ and $b_i = \int_0^1 \ell_i(\tau) d\tau$, for $i, j = 1, \ldots, s$, are equivalent to $C(q)$ and $B(q)$ with $q = s$.

Exercise 3. Consider a Runge-Kutta method characterized by the coefficients (a_{ij}, b_i, c_i), $i, j = 1, \ldots, s$. Let $0 \leq c_2 < c_3 < \cdots < c_{s-1} \leq 1$ and b_1, b_s be given. Further, assume that $c_1 = 0, c_s = 1$ and $a_{i1} = b_1, a_{is} = 0$, for $i = 1, \ldots, s$. Show that the remaining coefficients are uniquely determined by $C(s-2), B(s-2)$.