Series 3

Consider the Cauchy problem

$$y' = f(t, y), \quad y \in \mathbb{R}^n,$$
$$y(t_0) = y_0.$$

Definition : Given $s \in \mathbb{N}^*$ *collocation points* $0 \leq c_1 < c_2 < \ldots < c_s \leq 1$, we construct a polynomial $u(t)$ of degree s such that

$$u(t_0) = y_0$$
$$u'(t_0 + c_i h) = f(t_0 + c_i h, u(t_0 + c_i h)), \qquad i = 1, \ldots, s.$$

The *collocation method* is then defined by setting $y_1 = u(t_0 + h)$.

We will see in the lecture the following result :

Theorem : The collocation method is equivalent to the s-stage Runge–Kutta method with coefficients given by

$$a_{ij} = \int_0^{c_i} \ell_j(\tau)\, d\tau, \qquad b_i = \int_0^1 \ell_i(\tau)\, d\tau,$$

where the ℓ_i are the Lagrange interpolant polynomials given by

$$\ell_i(\tau) = \prod_{i \neq k} \frac{\tau - c_k}{c_i - c_k}.$$

Exercise 1. (Quadrature formulas and collocation methods) Let $0 \leq c_1 < \ldots < c_s \leq 1$ be given. Consider for $a_{ij}, b_j \in \mathbb{R}$, for $i, j = 1, \ldots, s$, the relations

$$C(q): \qquad \sum_{j=1}^{s} a_{ij} c_j^{k-1} = \frac{c_i^k}{k}, \quad k = 1, \ldots, q,$$
$$B(q): \qquad \sum_{i=1}^{s} b_i c_i^{k-1} = \frac{1}{k}, \qquad k = 1, \ldots, q.$$

i) Show that $C(q)$ for $q = s$ uniquely determines $a_{ij}, i, j = 1, \ldots, s$. Similarly, show that $B(q)$ for $q = s$ uniquely determines $b_i, i = 1, \ldots, s$.

ii) Show that if $C(q)$ holds for $q = s$ then $\int_0^{c_i} p(\tau)d\tau = \sum_{j=1}^s a_{ij} p(c_j)$ for all polynomials p of degree $\deg(p) \leq s - 1$. Similarly, show that if $B(q)$ holds for $q = s$ then $\int_0^1 p(\tau)d\tau = \sum_{i=1}^s b_i p(c_i)$ for all polynomials p of degree $\deg(p) \leq s - 1$.

iii) In particular: Show that a collocation method is consistent (i.e. $\sum_{i=1}^s b_i = 1$) and invariant under transformation into autonomous form.

Exercise 2. Compute the Runge–Kutta coefficients of all collocation methods with $s = 2$ nodes as a function of the nodes c_1 and c_2. Give the order of the methods.

Exercise 3. (Gauss, Radau and Lobatto quadrature) Let $0 \leq c_1 < \ldots < c_s \leq 1$ and consider the quadrature formula

$$\int_0^1 f(x)dx \approx \sum_{i=1}^{s} b_i f(c_i).$$

i) Show that there exist unique scalars $b_1, \dots, b_s$ such that the above quadrature formula has order (at least) s, i.e.,

$$\int_0^1 p(x)\mathrm{d}x = \sum_{i=1}^{s} b_i p(c_i) \quad \forall p \in \mathbb{P}^{s-1},$$

where $\mathbb{P}^{s-1}$ is the set of polynomial of degree $s-1$.

ii) Consider the Gauss nodes $c_1, \dots, c_s$ which are the zeros of the shifted Legendre polynomial

$$\frac{d^s}{dx^s}(x^s(1-x)^s). \tag{1}$$

First, show that the c_i's are distinct and lie in the open interval $(0,1)$. Then, prove that the associated Gauss quadrature formula has order $2s$ and the corresponding b_i's are positive.

iii) Consider the Radau nodes $c_1, \dots, c_s$, the zeros of

$$\frac{d^{s-1}}{dx^{s-1}}(x^{s-1}(1-x)^s). \tag{2}$$

Show that the c_i's are distinct (observe that $c_s = 1$) and lie in the interval $(0,1]$, that the associated Radau quadrature formula has order $2s-1$ and the corresponding b_i's are positive.

iv) Consider the Lobatto nodes $c_1, \dots, c_s$, the zeros of

$$\frac{d^{s-2}}{dx^{s-2}}(x^{s-1}(1-x)^{s-1}). \tag{3}$$

Show that the c_i's are distinct (observe that $c_1 = 0, c_s = 1$) and lie in the closed interval $[0,1]$, that the associated Lobatto quadrature formula has order $2s-2$ and the corresponding b_i's are positive.

Hint.

- The polynomials (1), (2) and (3) are related to Jacobi orthogonal polynomials on $[-1,1]$ which are defined for $n \in \mathbb{N}$, $\alpha, \beta > -1$ by

$$p_n^{(\alpha,\beta)}(x) = \frac{1}{e_n^{(\alpha,\beta)} W(x)} \frac{d^n}{dx^n}\big(W(x)(1+x)^n(1-x)^n\big), \quad \text{where} \quad W(x) = (1-x)^\alpha(1+x)^\beta,$$

  and the coefficients $e_n^{(\alpha,\beta)}$ are some renormalization scalars. The family of polynomials $\{p_n^{(\alpha,\beta)} \,|\, n \geq 0\}$ is orthogonal with respect to the scalar product with weight $W(x)$

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)W(x)dx.$$

  Then, one can consider the shifted Jacobi orthogonal polynomials $p_n^{(\alpha,\beta)}(2x-1)$ which are orthogonal on the interval $(0,1)$ with respect to the shifted weight $w(x) = W(2x-1)$.

- To show the order of the quadrature formulas, let $f(x)$ be a polynomial of degree $2s-1$, $2s-2$, $2s-3$, respectively, and perform an Euclidian division of $f(x)$ by appropriate polynomials.

Exercise 4. (Resolvent)

Consider the system of differential equations in dimension 2,

$$y'(t) = \begin{pmatrix} t & 1 \\ 0 & 0 \end{pmatrix} y(t), \qquad y(t_0) = y_0.$$

i) Compute the resolvent $R(t,t_0)$.

Recall: The columns $R_i(t)$ of the 2×2 matrix $R(t,t_0)$ are defined as the solution of $R_i'(t) = AR_i(t),\ R_i(t_0) = e_i$, where (e_1,e_2) is the canonical basis of $\mathbb{R}^2$.

ii) Compute the matrix

$$e^{\int_{t_0}^t A(s)ds},$$

and show that

$$y(t) = R(t,t_0)y_0 \neq e^{\int_{t_0}^t A(s)ds} y_0.$$