Series 2

Exercise 1.

i) By applying the mean value theorem to $\int_{t_0}^{t_0+h} f(s, y(s))\mathrm{d}s$, motivate the θ method

$$y_1 = y_0 + hf\Big(t_0 + \theta h, y_0 + \theta(y_1 - y_0)\Big), \tag{1}$$

for the approximation of the initial value problem

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0.$$

Which methods do we obtain for $\theta = 0$, $\theta = \frac{1}{2}$, and $\theta = 1$?

ii) Further, another variant of the θ method is given by

$$y_1 = y_0 + h(1-\theta)f(t_0, y_0) + h\theta f(t_0 + h, y_1). \tag{2}$$

Which methods do we obtain for $\theta = 0$, $\theta = \frac{1}{2}$, and $\theta = 1$?

iii) Show that both versions (1) and (2) are Runge-Kutta methods (give the coefficients).

Exercise 2. Give the order of the methods defined by (1) and (2) in Exercise 1 for the values $\theta = 0$, $\theta = \frac{1}{2}$, and $\theta = 1$. Further, show that the Runge method (see course) has order 2.

Exercise 3. Show that the order p of an s-stage explicit Runge-Kutta method satisfies $p \leq s$.

Exercise 4. Derive the order conditions for the coefficients a_{ij}, b_i, for $i, j = 1, \ldots, s$, of a Runge-Kutta method such that the method has order $p = 3$.

Exercise 5. Show that, if the field $f(t, y)$ satisfies a Lipschitz condition in y, an implicit Runge-Kutta method applied to $y'(t) = f(t, y(t))$ is well defined (for a stepsize that is small enough), i.e., the nonlinear system defining the method has a unique solution.
Hint : use a fixed-point theorem.

Exercise 6. Consider a Runge-Kutta method with coefficients a_{ij}, b_i, for $i, j = 1, \ldots, s$. Suppose that the method has order p for all autonomous systems

$$y'(t) = f(y(t)), \qquad y(0) = y_0.$$

Show that, if the coefficients c_i are defined by $c_i = \sum_{j=1}^{s} a_{ij}$, for $i = 1, \ldots, s$, this method has also order p when applied to a non autonomous system

$$y'(t) = f(t, y(t)), \qquad y(0) = y_0.$$

Recall. For a consistent method, i.e., with order $p \geq 1$, we have $\sum_{i=1}^{s} b_i = 1$.
Hint : use the transformation $Y(t) = \begin{pmatrix} t \\ y(t) \end{pmatrix}$

General information and series on `http://anmc.epfl.ch/Numerical.html`