Series 13

Stabilized explicit methods for stiff ODEs

Exercise 1.

i) Show that "the" Runge–Kutta method

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 2/6 & 2/6 & 1/6 \end{array}$$

is of order 4 and that it does not admit an embedded formula of order 3.

ii) Prove that in general, no 4-stage Runge–Kutta method of order 4, $y_1 = y_0 + h\sum_{i=1}^{s} b_i k_i$, admits an embedded formula of order 3, $\hat{y}_1 = y_0 + h\sum_{i=1}^{4} \hat{b}_i k_i$.

Hint for *ii)* : Define $d_j = b_j - \hat{b}_j$ and show that $d_j = 0$ for $1 \le j \le 4$ (which is a contradiction). To do so, define the matrices

$$U = \begin{pmatrix} b_2 & b_3 & b_4 \\ b_2c_2 & b_3c_3 & b_4c_4 \\ d_2 & d_3 & d_4 \end{pmatrix}, \qquad V = \begin{pmatrix} c_2 & c_2^2 & \sum_{j=1}^4 a_{2j}c_j - c_2^2/2 \\ c_3 & c_3^2 & \sum_{j=1}^4 a_{3j}c_j - c_3^2/2 \\ c_4 & c_4^2 & \sum_{j=1}^4 a_{4j}c_j - c_4^2/2 \end{pmatrix},$$

compute UV and conclude using the order conditions.

Exercise 2. **(Chebyshev polynomials)** For $s \in \mathbb{N}$ and $x \in [-1,1]$ we define

$$T_s(x) = \cos(s \arccos x).$$

By setting $x = \cos\varphi$, $\arccos x = \varphi$ show that $T_s(x)$ satisfy the following properties.

i) For $s = 0, 1$ we have $T_0(x) = 1$, $T_1(x) = x$ and further

$$T_s(x) = 2x\,T_{s-1}(x) - T_{s-2}(x) \qquad s \ge 2. \tag{1}$$

Remark: Thus, $T_s(x)$ is a polynomial of degree s with leading term $2^{s-1}x^s$. Moreover, the functions $T_s(x)$ can now be defined on $\mathbb{R}$ by virtue of (1) (and not only $[-1,1]$).

ii) Local extrema: $T_s\left(\cos\left(\frac{k\pi}{s}\right)\right) = (-1)^k, \quad k = 0, 1, \ldots s,$
Zeros: $T_s\left(\cos\left(\frac{(2k+1)\pi}{2s}\right)\right) = 0, \quad k = 0, \ldots s-1.$

iii) The polynomials $T_s(x)$ are orthogonal on $[-1,1]$ wrt. the weight function $\frac{1}{\sqrt{1-x^2}}$.

iv) Derivatives at $x = 1$: $T_s'(1) = s^2$, $T_s''(1) = \frac{1}{3}s^2(s^2-1)$.

Exercise 3. (Runge–Kutta Chebyshev method) For $\dot{y} = f(y)$ with $y(0) = y_0$, we consider the s-stage Chebyshev method defined by

$$\begin{cases} g_0 = y_0, \\ g_1 = y_0 + \frac{\Delta t}{s^2} f(g_0), \\ g_i = \frac{2\Delta t}{s^2} f(g_{i-1}) + 2g_{i-1} - g_{i-2} \qquad i = 2, \ldots, s, \\ y_1 = g_s. \end{cases} \tag{2}$$

i) Show that the method (2) applied to $\dot{y} = \lambda y$, $y(0) = y_0$ gives after one step

$$y_1 = T_s\left(1 + \frac{\Delta t \lambda}{s^2}\right) y_0,$$

where $T_s(x)$ is the Chebyshev polynomial of degree s.

ii) Show that the Runge–Kutta method (2) has order $p = 1$.

iii) If we apply the method (2) to the non-autonomous system

$$\dot{y} = f(t, y), \quad y(t_0) = y_0,$$

how should we update the time $t_0 + c_i \Delta t$ at the intermediate stages?

Exercise 4. (Stability analysis of the Runge–Kutta Chebyshev method)

Consider the Runge–Kutta Chebyshev method (2) applied to $\dot{y} = \lambda y$ with $\lambda \in \mathbb{R}^-$ and $|\lambda|$ possibly large.

i) What is the stepsize restriction in function of the number of stages s?

ii) Show that by varying s one can obtain an unconditionally stable explicit Runge–Kutta method.

iii) Based on the linear stability analysis compare the cost of the Runge–Kutta Chebyshev method and the explicit Euler method. (The cost is defined as the number of function evaluations.)

Remark. The stability function of the Runge–Kutta Chebyshev method $T_s\left(1 + \frac{\Delta t \lambda}{s^2}\right)$ equioscillates between $+1$ and -1 on the real axis. Let $\omega_0 = 1 + \frac{\eta}{s^2}$ and $\omega_1 = \frac{T_s(\omega_0)}{T_s'(\omega_0)}$ for some damping parameter η. Then the function given by $R_s^\eta(z) := \frac{T_s(\omega_0 + \omega_1 z)}{T_s(\omega_0)}$ equioscillates between $+\mu$ and $-\mu$ for some $\mu < 1$. The latter is the stability function of the so-called Runge–Kutta Chebyshev method with damping, which includes a strip around the negative real axis in its stability domain.

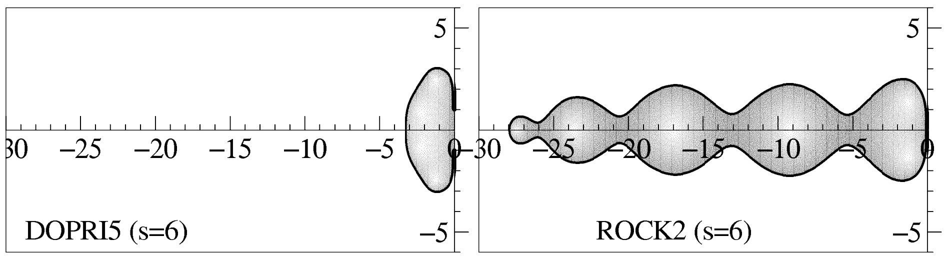

Figure 1: Comparison of stability domain for DOPRI5 and for ROCK2 (second order Chebyshev method; same numerical work as DOPRI5 for $s = 6$ stages).

General information and series on `http://anmc.epfl.ch/Numerical.html`