Series 12

Exercise 1. Consider a collocation method with $s = 2$ nodes. Discuss for which values of c_1, c_2 the method is A-stable.

Exercise 2. Consider the linear system $y' = Ay$, where A is a complex $n \times n$ matrix. We assume that

$$\mathrm{Re}\,\langle y, Ay\rangle \le 0 \qquad \forall\, y \in \mathbb{C}^n, \tag{1}$$

where $\langle \cdot, \cdot\rangle$ denotes the canonical scalar product in $\mathbb{C}^n$.

i) Show that any solution of $y' = Ay$ has the property that $\|y\|$ is a decreasing function in time.

ii) Let $R(z)$ be a rational function that is bounded for $\mathrm{Re}\, z \le 0$. Prove that in the norm

$$\|C\| = \sup_{\substack{u,v\in\mathbb{C}^n \\ \|u\|\le 1, \|v\|\le 1}} |\langle u, Cv\rangle| \qquad \text{for } C \in \mathbb{C}^{n\times n},$$

i.e., in the matrix norm associated to the scalar product, we have

$$\|R(A)\| \le \sup_{\mathrm{Re}\, z\le 0} |R(z)|.$$

iii) Let $R(z)$ be the stability function of an A-stable Runge–Kutta method. Deduce that the numerical solution $y_{n+1} = R(hA)y_n$ is contractive

$$\|y_{n+1}\| \le \|y_n\|.$$

Hints for ii).

a) Consider first the case where A is a normal matrix, i.e., A satisfies $AA^* = A^*A$. Recall, that a normal matrix can be diagonalized by a unitary matrix Q, i.e., $QQ^* = Q^*Q = I$.

b) For a general matrix A, consider the matrix function

$$A(w) = \frac{w}{2}(A + A^*) + \frac{1}{2}(A - A^*).$$

Show that $A(w)$ satisfies (1) for all $\mathrm{Re}\, w \ge 0$.

c) Show that for fixed vectors $u, v \in \mathbb{C}^n$ the rational function $\varphi(w) = \langle u, R(A(w))v\rangle$ has no poles in $\mathrm{Re}\, w \ge 0$.

d) Show that $A(iy)$ is a normal matrix for any $y \in \mathbb{R}$.

e) Prove that $|\varphi(1)| \le \sup_{y\in\mathbb{R}} |\varphi(iy)| \le \sup_{\mathrm{Re}\, z\le 0} |R(z)|\, \|u\|\|v\|$.

Exercise 3. Let $(k, j) \in \mathbb{N} \times \mathbb{N}$ be given.

i) Show that there exists a unique rational approximation $\frac{P(z)}{Q(z)}$ to the exponential e^z of order $k + j$ with $\deg(P) = k$, $\deg(Q) = j$ and $Q(0) = 1$.

ii) Prove that the rational approximation $R(z)$ studied in *i*) is explicitly given by $R_{kj}(z) = \frac{P_{kj}(z)}{Q_{kj}(z)}$ with

$$\begin{aligned} P_{kj}(z) &= 1 + \frac{kz}{j+k} + \ldots + \frac{k(k-1)\cdots 1}{(j+k)\cdots(j+1)}\frac{z^k}{k!}, \\ Q_{kj}(z) &= 1 - \frac{jz}{k+j} + \ldots + (-1)^j \frac{j(j-1)\cdots 1}{(k+j)\cdots(k+1)}\frac{z^j}{j!}. \end{aligned}$$

Hint: Use Lemma 1 and Lemma 2 of the Chapter *"Stability domains of collocation methods"* of the course. Those provide results for rational functions of the type

$$R(z) = \frac{M^{(s)}(1) + zM^{(s-1)}(1) + \cdots + z^s M(1)}{M^{(s)}(0) + zM^{(s-1)}(0) + \cdots + z^s M(0)},$$

where $M(t)$ is a polynomial of degree s and satisfies $M^{(s)}(t) = 1$. Try to find an appropriate polynomial $M(t)$ such that the numerator has degree k and the denominator has degree j.

Exercise 4. Consider a numerical method Φ_h of order p for systems of differential equations

$$\dot{y} = f(y), \quad y(t_0) = y_0.$$

Let $y_2 = \Phi_h \circ \Phi_h(y_0)$ and $\omega = \Phi_{2h}(y_0)$. Show that

$$y(t_0 + 2h) - y_2 = 2C(y_0)h^{p+1} + \mathcal{O}(h^{p+2}), \tag{2}$$

$$y(t_0 + 2h) - \omega = C(y_0)(2h)^{p+1} + \mathcal{O}(h^{p+2}), \tag{3}$$

where the same quantity $C(y_0)$ appears in both equations (2) and (3).