Series 10

Exercise 1. (Curtis and Hirschfelder, 1952)
Consider the following problem

$$y'(t) \;=\; \lambda(y(t) - \cos t), \qquad y(0) = y_0, \tag{1}$$

where $\lambda < 0$.

i) Compute the exact solution of (1).
ii) Perform a linear stability analysis and report the time-step restriction for the explicit Euler method applied to the linearized problem. What happens if $|\lambda|$ becomes large?
iii) What is the time-step restriction for the implicit Euler method?

Exercise 2. For an initial value problem $y' = f(t, y),\ y(0) = y_0$, $(y(t) \in \mathbb{R}^d)$ we consider the trapezoidal method

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right). \tag{2}$$

What is the time-step restriction for the method (2) applied to the linearized problem obtained from (1)?

Exercise 3. (Eigenvalues of the discrete Laplacian) Consider the following $N \times N$ matrix, used for the discretization of the Laplacian in space-dimension one with Dirichlet conditions,

$$A = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & & \end{pmatrix}.$$

Show that the eigenvalues of A are

$$\lambda_k = -2 + 2\cos\left(\frac{k\pi}{N+1}\right), \qquad 1 \le k \le N,$$

with associated eigenvectors $v^{(k)} = \left(v_1^{(k)}, \dots, v_N^{(k)}\right)^T$ given by

$$v_j^{(k)} = C \sin\left(\frac{jk\pi}{N+1}\right), \qquad 1 \le j, k \le N.$$

Hint: Notice that an eigenvalue μ of $B := A + 2I$ and the components of the associated eigenvector $v = (v_1, \dots, v_N)^T$ satisfy the recurrence relation

$$v_0 = 0, \qquad v_{j-1} + v_{j+1} = \mu v_j, \quad j = 1, 2, \dots, N, \qquad v_{N+1} = 0.$$

Using the ansatz $v_j = \zeta^j$ deduce the relation $v_j = C\left(\zeta_1^j - \zeta_2^j\right)$, where $\zeta_1, \zeta_2 \in \mathbb{C}$ satisfy

$$\zeta_1 + \zeta_2 = \mu, \quad \zeta_1\zeta_2 = 1 \quad \text{and} \quad \left(\frac{\zeta_1}{\zeta_2}\right)^{N+1} = 1.$$

Then, find the values of ζ_1, ζ_2 and μ.

Exercise 4. Consider the Brusselator problem, for $x \in (0,1),\ t \geq 0$,

$$\frac{\partial u}{\partial t} = a + u^2 v - (b+1)u + \nu \frac{\partial^2 u}{\partial x^2},$$
$$\frac{\partial v}{\partial t} = bu - u^2 v + \nu \frac{\partial^2 v}{\partial x^2}.$$

Take the boundary conditions

$$u(0,t) = u(1,t) = 1, \qquad v(0,t) = v(1,t) = 3$$

and initial conditions

$$u(x,0) = 1 + \sin(2\pi x), \qquad v(x,0) = 3.$$

Give explicitly the system of ODEs obtained by discretizing the spatial variable of the above system applying the method of lines (see lecture). Using Exercise 3, perform a linear stability analysis of the resulting system of ODEs and give the time-step restriction for the explicit Euler method.

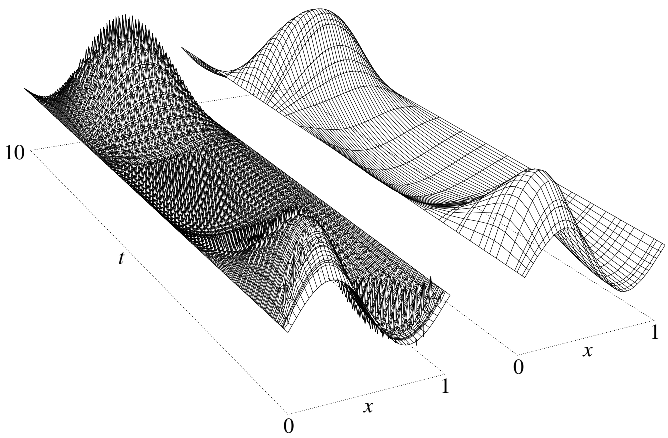

Figure 1: Numerical solutions (using DOPRI5 and ROCK2) of the Brusselator problem for substance $u(x,t)$ with parameters $a = 1$, $b = 3$ and $\nu = 0.1$ (Exercise 4).

Exercise 5. Let $\Omega \subset \mathbb{R}$ be an open bounded subset and u be a smooth function $u : \Omega \longrightarrow \mathbb{R}$. Further, let $x_i \in \Omega$ and $\Delta x > 0$ such that $[x_{i-1}, x_{i+1}] \subset \Omega$ with $x_{i\pm1}$ given by $x_{i\pm1} = x_i \pm \Delta x$.

i) Show that

$$\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{(\Delta x)^2} = \frac{d^2 u}{dx^2}(x_i) + \mathcal{O}\left((\Delta x)^2\right).$$

ii) How many time should u be differentiable for the proof of *i)*?

iii) What happens to the above approximation of $\frac{d^2u}{dx^2}(x_i)$ if u is only $u \in \mathcal{C}^2(\Omega)$, i.e., twice continuously differentiable?

General information and series on `http://anmc.epfl.ch/Numerical.html`